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The Journey Continues: Mathematics Curriculum Analysis from the Official Curriculum to the Intended Curriculum

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ABSTRACT

In this paper, we share major lessons we have learned in our curriculum analysis explorations and provide suggestions for future mathematics curriculum research. Specifically, we will discuss the successes and challenges of using comparative methods and developing analytical frameworks. Using an old Chinese saying, follow the vine to get the melon (顺藤摸瓜), our journey starts with getting a holistic picture of standards and moves to identifying a specific topic in the textbooks. We first conducted a study to compare the geometry standards of the Common Core State Standards for Mathematics (CCSSM) and the Chinese Compulsory Education Mathematics Curriculum Standards (CMCS); then we analyzed the presentation of a specific topic, triangle congruence, in multiple geometry textbooks. For each comparative study we conducted, developing the analytical framework that can be used to guide future investigations constituted a critical step in the research methodology. We have learned lessons from adapting the well-known van Hiele model, which was created for the development of geometric reasoning, into a lens for a detailed curriculum analysis. We have also learned lessons from elaborating on the key constructs from the "Mathematics Curriculum as Story" framework for coding and analysis. These lessons as well as our future directions serve as the primary focus of this paper.

KEYWORDS

Comparative study; geometry; mathematics curriculum as story; van Hiele levels.

INTRODUCTION

Since 2018, our research team has been engaged in curriculum studies comparing and analyzing U.S. and Chinese mathematics curricula regarding geometry, including curriculum standards and textbooks. As mathematics educators, we all have completed our K–12 education under the Chinese educational system and teach K–12 preservice teachers in the U.S. The cross-national learning and teaching experiences provided us firsthand experiences with mathematics curricula in both China and the U.S. We naturally engaged in curriculum comparative studies in the two countries. This paper intends to convey the insights we have gained about methodology and framework during this journey, with the goal of inspiring future undertakings in curriculum studies.

Geometry curriculum drew our attention because we have learned from cross-national assessments (e.g., TIMSS) that U.S students' geometry performance is far behind their counterparts from East Asian countries (Mullis et al., 2015). Compared with other areas of mathematics, U.S. geometry curriculum is relatively weak, which leads to students' less competitive achievements (e.g., Wu & Wang, 2022). In addition, we have noticed from our teaching that many preservice teachers in the U.S. have less interest in geometry and are afraid of teaching geometry. We navigated our research topic in geometry and started with standards comparison. As official curriculum, standards determine learning goals and provide a blueprint for teacher-intended curriculum (e.g., standards-aligned textbooks), and then potentially influence assessments that measure how learning goals have been achieved (Remillard & Heck, 2014). Starting from the standards, we intended to gain an overall understanding of the geometry curriculum in China and the U.S. The standards distribute the mathematics content into yearly learning goals, which raised our interest in what content students are provided and how students are expected to learn year by year.

With these initial inquiries, we chose to investigate the Common Core State Standards for Mathematics (CCSSM), which is a commonly adopted designated curriculum across the states (Reys, 2014). Compared to pre-CCSSM standards, CCSSM made a considerable change in geometry; it places great emphasis on geometric concepts and informal and formal deduction, and increases attention on transformational geometry, including the verification of congruence and similarity through a series of transformations (e.g., Conley et al., 2011; Dingman et al., 2013; Tran et al., 2014). In addition, CCSSM is a set of learning goals that was developed as a result of the international assessments and cross-national curriculum comparisons (e.g., Schmidt et al., 2005) to address the "mile wide and an inch deep" characteristics of the U.S. curriculum. Compared with the decentralized U.S educational system, the Chinese education system is highly centralized; since its release, the Chinese Compulsory Education Mathematics Curriculum Standards (CMCS) has been the only up-to-date set of national standards (Ministry of Education, 2011). Using a comparative methodology, we compared CCSSM and CMCS and obtained a comprehensive understanding of how these two sets of standards are organized, how they proceed with big geometrical ideas, what learning goals are created by schooling years, and what guidance is offered by standards toward addressing these goals (Lo et al., 2022).

Keeping in mind the idea that standards are "ends" and textbooks are "means" (Stein et al., 2007), one of the outstanding findings from the comparison between CCSSM and CMCS was that the two official curricula present the topic of *congruence* differently. This finding from the standards analysis prompted us to take a closer look at how similarly or differently textbooks present the topic of congruence based on these two approaches. To achieve this goal, we conducted textbook analyses of the topic of triangle congruence in which we compared several American textbooks and a Chinese textbook (Lo et al., 2021; 2023). The standards analysis built a foundation for our subsequent textbook analysis. From the standards analysis, we mapped the concept of congruence vertically, i.e., how the curriculum builds mathematical understanding of the concept across grade levels, how the concept is developed over the years, and how the concept is extended to other new concepts. We also charted the concept horizontally by exploring related mathematical ideas and concepts in the same grade level. Through our exploration, we have developed two distinct perspectives. The first perspective focuses on the development of particular concepts that align with the learning objectives defined in the standards. The second perspective involves comprehensively understanding specific mathematical concepts in textbooks.

Like conducting research in any area, in this process we have learned from our own experiences of the inquiry as well as the experiences of others. Our journey in curriculum studies, from standards to textbooks, can be described by an old Chinese saying, *Follow the vine to get the melon* (*顺藤摸瓜*), which means first getting a holistic picture of the official curriculum standards, and then moving on to a specific topic in the textbooks. As our research team reflected on the journey in curriculum studies from standards to textbooks and envisioned future directions, two key lessons stood out. Beyond sharing our research journey in curriculum study, this paper reports on the key lessons we have learned: (1) In curriculum studies, even the accepted theories or frameworks may not be applicable for some specific topics; researchers thus need to adapt theories or develop frameworks aligning with the focus of the analysis; (2) Comparative methodological tools are an effective approach to conducting curriculum analysis, serving as a mirror that allows us to discover nuanced information that might be neglected when focusing on only one curriculum. Along with the lessons learned, we also discuss factors that we have yet to capture in our current analyses. Finally, we look forward to future directions for continuing this journey of curriculum analyses.

Lesson # 1. Adapting Framework Aligning with the Focus of the Analysis

In curriculum studies, even the accepted theories or frameworks might not be directly applicable to some specific research questions; researchers thus need to adapt theories or develop frameworks aligning with the focus of their analysis. For each study we conducted, developing the analytical framework that could be used to guide future investigations constituted a critical step for the research methodology. We have learned lessons from adapting the well-known van Hiele model, which was created for the development of geometric reasoning, into a lens for a detailed curriculum standards analysis. We have also learned lessons from elaborating the key constructs from the "Mathematics Curriculum as Story" framework through a narrative lens to gain a holistic view of textbooks. In this section, we use specific examples to illustrate our adaption of these two frameworks in our studies.

Adapting the van Hiele Level Framework

With the purpose of examining the expectations of geometrical development, we needed a framework that could provide theoretical support to analyze the levels of learning experiences in standards. We also needed the framework to support our analysis focusing on the learning trajectory of specific topics within and across grade levels. van Hiele levels (1959/1984), rooted in mathematics education, is a frequently used framework to describe students' geometric development. Though discussions around whether the van Hiele levels are sequential, linear, or discrete have been continuing, previous research has confirmed that the van Hiele levels model is an effective tool to examine students' geometrical understanding (e.g., Fuys et al., 1988; Senk, 1989; Usiskin, 1982; Wang & Kinzel, 2014). Specifically, researchers have used van Heile levels to compare learning expectations in different standards and have confirmed van Hiele levels could be used as a framework to examine the quality of geometry standards (e.g., Dingman et al., 2013; Newton, 2011).

Initial Use of the Existing Framework

Through sufficient literature review, we confirmed that the van Hiele theory is applicable as an analytic framework for geometry standards comparison. However, when we went into the analysis process, we found the levels were too broad and not sufficiently applicable to directly identify the level of a learning expectation. For instance, the feature of van Hiele level 2 is that "properties are perceived, but they are isolated and unrelated" (Mayberry, 1983, p. 59), and the feature of van Hiele level 3 is that "definitions are meaningful, with relationships being perceived between properties and between figures. Logical implications and class inclusions are understood. The role and significance of deduction, however, is not understood" (Mayberry, 1983, p. 59). We were not able to reach a consensus on some specifics of the coding when considering the textual meaning of standards.

Expanding the Framework

To increase the validity of coding, we used a direct comparison approach to carefully examine each learning expectation in the two documents (Tran et al., 2016). Drawing from Newton (2011) and Tran et al. (2014), we considered a learning expectation to be a standard unit or a component of a standard unit. If a standard included multiple mathematics foci, the standard was split into multiple learning expectations (Lo et al., 2022). In this phase, we looked for emerging themes. As we revisited the data multiple times, the learning expectations were sorted into topics, such as *Recognizing figures*, *Understanding relationships between 3D and 2D figures*, *Transformation*, etc. Under each topic, we noted learning expectations were in a progressive sequence across grade levels/bands in both standards documents. Then, we returned to the van Hiele levels framework and developed topics at each level (see Table 1 below).

Table 1.
Topics at van Hiele Levels

Van Hiele Levels	Topics
Level 1	Recognizing figures Composing and decomposing figures
Level 2	Recognizing figures by their specified attributes Understanding relationships between 3D and 2D figures empirically Recognizing transformation empirically Recognizing definitions empirically
Level 3	Classifying figures Understanding definitions Developing informal proofs Constructing figures Understanding transformation Understanding relationships between 3D and 2D figures
Level 4	Writing formal proofs

(Lo et al., 2022, p. 44)

The same topic develops across van Hiele levels. For instance, *Recognizing transformation* at van Hiele level 2 expects students to name different transformations from manipulations; at van Hiele level 3, students are expected to understand transformations by describing the relationships (points, sides, and angles) between the two shapes and recognize the line of symmetry, the center of rotation, and translation direction and distance. We coded each learning expectation into both topic and van Hiele levels.

Take the standard “Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures” (CCSSI, 2010, p. 32), for example. Through our analysis, we identified two distinct learning expectations: drawing these geometric elements and identifying them within two-dimensional shapes. This highlights that students are not merely expected to recognize the individual geometric figures but also need to know their attributes. Consequently, we classified this standard as meeting the criteria for van Hiele level 2. The progressive aspect of learning expectations in the same topic helped us to identify their corresponding van Hiele levels. In turn, the van Hiele levels framework also supported us in tracking the development of geometric topics. The topics and van Hiele levels were two dimensions for our coding, which not only increased validity and reliability but also enriched the van Hiele level framework in standards analysis. By considering both dimensions, we are able to offer a more nuanced understanding of the curriculum standards. This approach not only ensures a more accurate classification of standards according to van Hiele levels but also sheds light on the progression of learning objectives.

Adapting the Story Framework for Textbook Analysis

As mentioned earlier, one of the main findings of the comparative standards analysis was the different treatments of the concept of triangle congruence in the U.S. and Chinese standards. We found that CCSSM defines congruence from a geometric transformation perspective as "the second can be obtained from the first by a sequence of rotations, reflections, and translations" (CCSSI, 2010, p. 55). CMCS expects students to understand the congruent triangle from the quantitative perspective, that corresponding pairs of sides and angles are congruent. This finding prompted us to take an in-depth look at the development of triangle congruence in two different mathematics textbooks: one from China and one from the U.S.

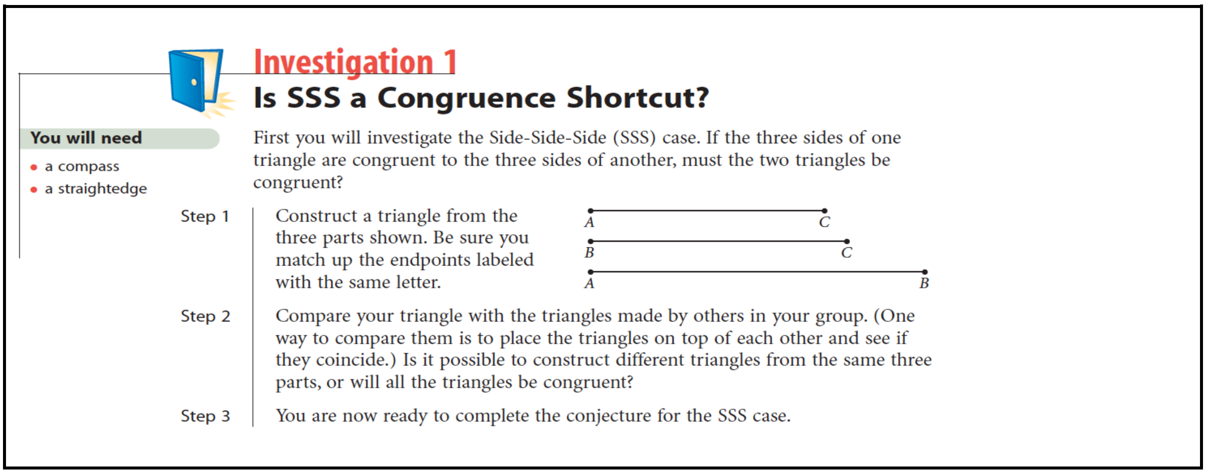
Research on mathematics textbooks has made significant contributions in identifying and reporting features that can be used as the basis for comparisons. However, these findings collectively still fail to provide a sense of the changes and flow of the mathematical ideas throughout a textbook, nor are they able to account for the aesthetic of sequencing and presenting ideas in one way or another (Dietiker, 2015). Recognizing this limitation of the current research on mathematics textbook analysis, Dietiker (2015) proposed a story framework to analyze textbooks that aimed at filling these gaps. The Mathematics Curriculum as a Story framework includes four components: characters, actions, setting, and plot. Mathematical characters are "figures" that were brought into existence by descriptive naming that could be in a variety of forms, such as words, graphs, tables, or symbolic forms. Mathematical actions are manipulations taken by a mathematical character that result in a mathematical change. Mathematical setting is the space in which the mathematical character emerged and developed via mathematical actions. A mathematical plot is the "potential temporal dynamics of the story that encourages (or discourages) a reader to continue to advance through the mathematical story" (Dietiker, 2015, p. 299). We will share the lessons we learned from this process when conducting comparative textbook analyses on lessons of triangle congruence using the "action" component for illustration. We chose this particular component because it was the most challenging component among the four in our attempts to operationalize the framework for textbook analysis. Focusing on it will illuminate the major lessons we learned.

Initial Identification of Actions

Dietiker (2015) defined mathematical actions as manipulations taken by an actor on a mathematical character that result in a mathematical change or creating a new mathematical character. The actors include the narrators or fictional characters embedded in the story or the readers, such as the researchers, teachers, or students. To identify the actions needed to make sense of triangle congruence, we started with a collection of triangle congruence units from several U.S. high school geometry textbooks. It is easier to identify the actions when the textbooks are written in the form of a dialogue between the authors and the readers. For example, Figure 1 shows an investigation of side-side-side congruence from Discovering Geometry (Serra, 2003), which uses the actions of (1) duplicating segments, (2) constructing a triangle from three segments, and (3) comparing your triangle with those made by others, to determine if two triangles are congruent—all made explicit by the text.

Figure 1.

An example of a geometric investigation (Serra, 2003, p. 220)



Investigation 1

Is SSS a Congruence Shortcut?

You will need
- a compass
- a straightedge

First you will investigate the Side-Side-Side (SSS) case. If the three sides of one triangle are congruent to the three sides of another, must the two triangles be congruent?

Step 1 Construct a triangle from the three parts shown. Be sure you match up the endpoints labeled with the same letter.

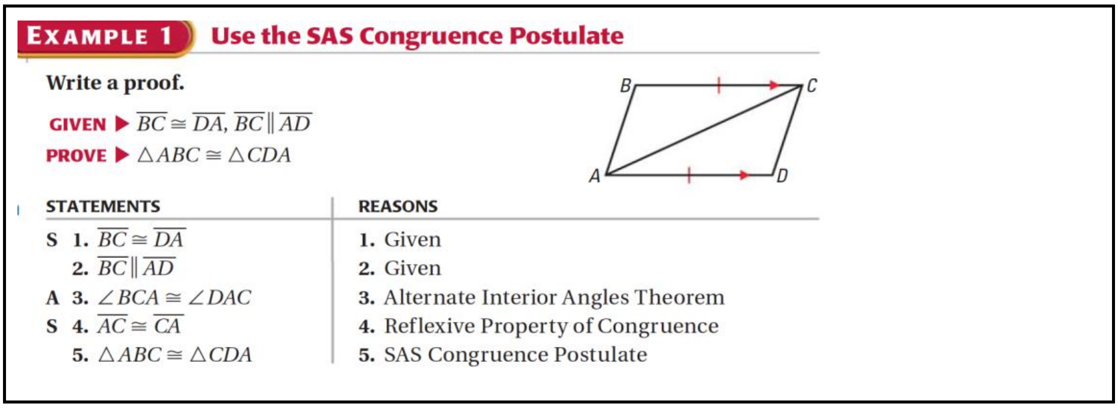
Step 2 Compare your triangle with the triangles made by others in your group. (One way to compare them is to place the triangles on top of each other and see if they coincide.) Is it possible to construct different triangles from the same three parts, or will all the triangles be congruent?

Step 3 You are now ready to complete the conjecture for the SSS case.

However, some textbooks contain worked-out examples with the solutions not written in a dialogue form, which is another common type of task in geometry textbooks. For example, Figure 2 contains an example of a deductive proof (Larson et al., 2007) that can be found in all geometry textbooks.

Figure 2.

An example of a deductive proof (Larson et. al. 2007, p. 240)



EXAMPLE 1 Use the SAS Congruence Postulate

Write a proof.

GIVEN ▶ $\overline{BC} \cong \overline{DA}$, $\overline{BC} \parallel \overline{AD}$

PROVE ▶ $\triangle ABC \cong \triangle CDA$

	STATEMENTS	REASONS
S	1. $\overline{BC} \cong \overline{DA}$	1. Given
	2. $\overline{BC} \parallel \overline{AD}$	2. Given
A	3. $\angle BCA \cong \angle DAC$	3. Alternate Interior Angles Theorem
S	4. $\overline{AC} \cong \overline{CA}$	4. Reflexive Property of Congruence
	5. $\triangle ABC \cong \triangle CDA$	5. SAS Congruence Postulate

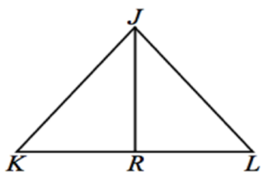
In this case, a thought experiment is conducted to identify the mental actions that are required to carry out such proof. Here, we identified one action of applying a previously established property in Step 3 when the alternate interior angles theorem was used to establish a pair of congruent angles. Furthermore, we identified another action of corresponding, which is needed to complete Step 5 when applying the SAS congruence correctly. Mathematically speaking, it would be incorrect to say triangle *ABC* is congruent to triangle *ADC*, even though we can name the same triangle with vertices in different consecutive order based on their locations: *ADC* or *CDA*. Therefore, a mental action is required to name the correct correspondence. Similar actions were identified from both investigative and proof types of activities from the geometry textbooks used for our analyses.

After we became more comfortable with identifying different actions needed for learning triangle congruence, we expanded our identification to more complex tasks from our collection of geometry textbooks. For example, we identified an additional action, translation, from the following exercise in *Eureka Math* (Great Minds, 2015) based on the solution provided in the teacher edition.

Figure 3.

An example of the "translation" code (Great Minds, 2015, p. 127)

Given: $JK = JL$; $\overline{JR}$ bisects $\overline{KL}$
Prove: $\overline{JR} \perp \overline{KL}$

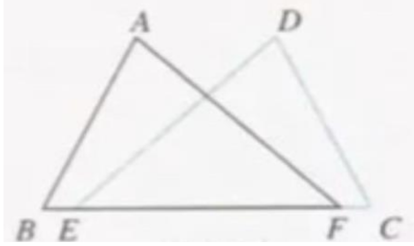


One needs to translate the statement "*JR* bisects *KL*" to a pair of congruent angles: $\angle KJR = \angle LJR$. Then the congruence between triangle *KJR* and triangle *LJR* can be established with SAS: *JK* = *JL* (S), $\angle KJR = \angle LJR$ (A), *JR* = *JR* (S). Since $\angle JRK$ and $\angle JRL$ are supplementary, $\angle JRK = \angle JRL = 90°$. This relationship needs to be translated back to the statement "*JR* ⊥ *KL*." This process of expansion continues until no new actions can be identified. To ensure the applicability of this work on comparative analyses of textbooks beyond those published in the U.S., we applied the identified action codes to a triangle congruence unit from a Chinese textbook. An additional action, "joining or separating," emerged from the following task (Figure 4) in the Chinese textbook (People's Education Press [PEP], 2013).

Figure 4.

An example of joining or separating (PEP, 2013, p. 39)

Given: The points *E* and *F* are on the segment *BC*,
BE = *CF*, *AB* = *DC*, ∠*B* = ∠*C*.
Prove: ∠*A* = ∠*D*



One approach to prove ∠A = ∠D is to prove that ΔABF is congruent to ΔDCE, which can be established with SAS. The problem statement already contains a pair of congruent sides,

$AB = DC$ (S), and one pair of congruent angles, $\angle B = \angle C$ (A). An additional pair of congruent sides, *BF* and *CE*, can be established by joining two segments to form BF = CE because $BF = BE + EF = CF + EF = CE$.

Revising the Action Codes

After we came up with a set of initial codes, we reviewed and revised them to make sure that the codes communicated well the intended meanings. For example, early in the analysis, we found that many tasks in the textbooks were presented with diagrams. Some diagrams contained two triangles that were displayed with the orientation in which the corresponding relationship could be identified directly, while other diagrams were much more complicated, like the ones in Figure 3 and Figure 4, where some types of mental transformations would be required to identify the corresponding relationship. Initially, we named this action “orientating” but found the meaning of this word too broad to communicate the intended meaning. Then we thought of the word “transforming.” However, we further revised this to “corresponding” actions, partly because it fits the nature of the action better and partly because we like to reserve the word “transforming” for the transformations of reflection, rotation, and translation formally introduced in some CCSSM-based textbooks such as Eureka Math. In the end, we identified eight different actions: (1) applying, (2) adding auxiliary lines, (3) constructing, (4) corresponding, (5) drawing and comparing, (6) joining or separating, (7) transforming, and (8) translating. Together with the additional analysis based on the other three components, the story framework was used to compare two congruent triangle units: one from the U.S. and one from China (Lo et al., 2023).

Looking back, the process of generating codes based on the existing theoretical framework for textbook analysis is iterative and ongoing. In the case of adapting the Story framework, we started with one textbook that was written in dialogue form, then moved to the other textbooks written in more formal mathematical language. We also started with coding simple tasks and then moved to more complex tasks. Finally, we applied the codes we generated from various U.S. geometry textbooks to a Chinese textbook to search for additional codes. We are open to and ready for the possibility that more codes will be needed when we continue to apply our codes to additional textbooks or standards from other countries included in our future analyses.

Lesson # 2. Comparative Methodological Tool Provides a Mirror for Curriculum Analysis

We are using the metaphor of a mirror to describe the comparative approach that we used in the curriculum analysis. Mirrors serve many functions in our daily life: We get face-to-face opportunities with ourselves, we see, and we learn about ourselves through the mirror. The mirror can also be used as an instrument to extend our vision and see the world in our blind spots. Like a mirror, comparative methods offer us a perspective that enables us to understand one curriculum better through learning about another.

We chose comparative methods in our curriculum studies partially because the cross-cultural comparison perspective offers us a unique opportunity to sensibly capture nuanced information, as confirmed by previous researchers (e.g., Cai, 2002, 2005; Ma, 1999). On the other hand, we also are conscious that familiarity with teaching and learning mathematics might increase the possibility of over-interpretation. Thus, comparative approaches help us to regulate objectivity and familiarity. We use the metaphor of a mirror to indicate the function of the comparison. Similar to Lesson #1, in this section we also use examples from our studies to illustrate the unique insights gained through comparative approaches.

Identifying Differences and Similarities Is Not the End Goal of Comparison

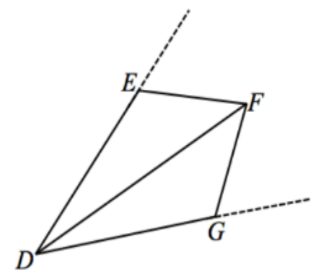
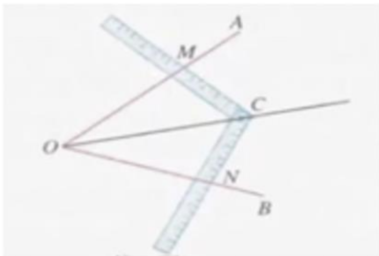
With the comparative methodological approach, we found differences and similarities are often intertwined; different approaches might lead to the same learning objectives, and similar approaches might serve different learning goals. For instance, in our standards comparison, we found that *transformation* in both documents consists of reflection, translation, rotation, and dilation, and both CCSSM and CMCS expect students to develop an understanding of transformations from concrete objects and empirical experiences. However, CMCS presents transformation crossing grade bands, which reflects curricular coherence, whereas CCSSM introduces transformation in grade 8, followed by the concepts of *congruence* and *similarity*, which are defined from a transformation perspective (Authors, 2022a). The same concept in the two documents serves different purposes but eventually expects students to achieve similar learning goals, that is, to recognize the corresponding points, sides, and angles with position changes. We infer that the same concept could be developed through multiple learning trajectories.

Through a comparative lens, we can capture new information from the curriculum we are familiar with. For instance, we did not realize that the connection between math and real life routinely provided in the PEP textbook was rather unique until we found that many geometry textbooks do not include applied mathematics in the topic of congruence. Figure 5 below shows two examples from Eureka Math and PEP Math with two triangles of a shared side. The given statements and the conclusions that can be proved in the two exercises are the same. The exercise from Eureka Math is a decontextualized problem in which the given two pairs of sides are congruent, which is directly provided. The exercise from PEP Math is a problem related to real life that describes a craftsmen's tool and provides a detailed description of how to use the tool. Students need to read, interpret the context, and abstract mathematical relations from words.

In PEP Math, the statement "ensure the measurements on the two points *M* and *N* stay the same" indicates $MC = NC$. Students need to capture the information and successfully transfer the information to mathematical relations to solve the problem. Noticing this difference prompted us to ask the following questions: "What learning opportunities can real-life problems and decontextualized problems provide?" and "What obstacles would students face encountering these different contexts?"

Figure 5.

An example of the same mathematical ideas presented in different contexts

Opening Exercise	Exercise 2
Write a proof for the following question. Once done, compare your proof with a neighbor's. Given: *DE* = *DG*, *EF* = *GF* Prove: *DF* is the angle bisector of $\angle EDG$.  (Great Minds, 2015, p. S136)	Craftsmen often use an angle bisector tool to bisect an angle. The procedures are listed below: As the figure shows, $\angle AOB$ is any angle, mark *OM* = *ON* from *OA* and *OB*, respectively. Move the angle square and ensure the measurements on the two points *M* and *N* stay the same. The line that goes through *C* and *O* is the angle bisector of $\angle AOB$. Why?  (PEP, 2013, p. 37)

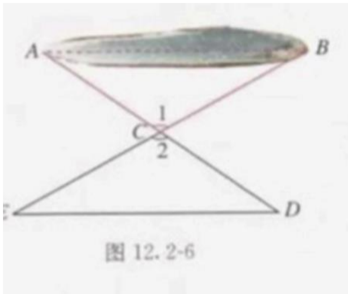
Enriching Understanding of Mathematics Curriculum Through Comparison

A comparative method is a powerful tool that continues to enrich our understanding of curricula and enables us to gain new information in the iterative process. As we continued to explore geometry textbooks through a comparative lens, we found similar approaches used across different textbooks to highlight the application of geometry. For instance, Figure 6 shows two exercises of measuring the width of a pond/stream, from Discovering Geometry and the PEP textbook, respectively, that connect the concepts of congruent triangles to real-life scenarios. The objective of the task—using a triangle congruent condition to measure a specific distance indirectly—is the same. However, two different triangle congruent conditions are used. ASA is used in the exercise from Discovering Geometry, while SAS is used in the exercise from PEP Math.

Furthermore, the nature of the presentation of each task is quite different. PEP Math describes the problem mathematically with mathematics relations such as *DE* = *CB* and *CD* = *CA* stated explicitly. It also provides students the opportunity to apply what they learned from the previous unit: that is, the vertical angles are congruent. Discovering Geometry offers a detailed description of a scenario and a diagram, yet all the mathematical relations are presented implicitly. Students need to identify the right angles from the context of holding a stick straight up, another pair of congruent angles from the action of "use the same line of sight," as well as keeping a firm grip on the pole to establish a pair of congruent side lengths from the ground to eye level. The problem in Discovering Geometry is more contextualized and requires students to correctly interpret the embedded mathematical relations.

Figure 6.

Examples of different mathematical approaches to the same real-life scenario

Exercise 15 Samantha is standing at the bank of a stream, wondering how wide the stream is. Remembering her geometry conjectures, she kneels down and holds her fishing pole perpendicular to the ground in front of her. She adjusts her hand on the pole so that she can see the opposite bank of the stream along her line of sight through her hand. She then turns, keeping a firm grip on the pole, and uses the same line of sight to spot a boulder on her side of the stream. She measures the distance to the boulder and concludes that this equals the distance across the stream. What triangle congruence shortcut is Samantha using? Explain.  Serra, 2003, p. 240	Example 2 To measure the distance between point *A* and point *B* outside a pool, select a point *C* on the land which can get to *A* and *B* without passing the pool. Draw the segment from *A* to *C* and extend it to *D* and make *CD*=*CA*. Draw the segment from *B* to *C* and extend it to *E* and make *CE*=*CB*. Draw the segment from *D* to *E*; the measurement of *DE* is equal to the distance from *A* to *B*. Why?  PEP, 2013, p. 38

The purpose of comparing is not to make claims about which curriculum is better. Merely reporting differences is insufficient to make a significant contribution to our ultimate goal of improving student learning. We need to understand the curriculum comprehensively as a system and use the findings to make suggestions. The comparative methodological approach provides us a lens to gain rich information to understand how the curriculum can support teachers/students to achieve learning objectives.

Looking Back and Looking Ahead

Reflecting on our journey of comparative curriculum analyses, we would like to acknowledge some potentially important factors that we have yet to fully consider in our current analyses. One such factor is the structural differences in the standards and textbooks from different countries. For example, CCSSM is organized by grade level through K–8 and by content-based standards in high school. CMCS is organized by three grade bands, Grades 1–3, Grades 4–6, and Grades 7–9. In order to compare, we decided to form comparable grade bands for CCSSM, as seen in Table 2. Although doing so makes the comparison possible, we might have failed to account for some intended connections across K–8 by the writers of CCSSM.

Table 2.

Grade Bands for Standards Analysis

	Lower-Grade Band	Middle-Grade Band	Upper-Grade Band
CCSSM	K–3	Grades 4–6	Grades 7, 8, & high school
CMCS	Grades 1–3	Grades 4–6	Grades 7–9

Similar examples can be seen in our study of the introduction of triangle congruence in U.S. and Chinese textbooks. For example, the topic of triangle congruence is taught in all Chinese schools in grade 8 as part of the compulsory education that has been prescribed by the national curriculum standards, CMCS. However, the same topic could be taught in any grade as part of the high school curriculum. The diverse high school student population might have played an important role in textbook design. Furthermore, the U.S. textbooks in our study are structured by lessons, while the Chinese textbooks are structured by main ideas in chapters and units without explicit demarcation of day-by-day lessons. By using lessons as the unit of comparison, we might have missed some important built-in continuity with the Chinese unit structure.

Another type of influencing factor is culturally rooted. For example, under the centrally controlled education system in China, geometry, as an important area of mathematics, is studied by all high school students. Furthermore, students' academic performance on the entrance exams determines their further education and even career selection. This kind of system directly affects teaching and learning, and thus impacts the written and intended curriculum. Prior research studies that have compared mathematics textbooks from the U.S. and several Asian countries, such as China, Japan, and Korea, which have examination-oriented systems, found that more mathematics topics are introduced consistently in earlier grades in the latter group than in the U.S. (e.g., Jones & Fujita, 2013; Wang et al., 2018). As an embodiment of culture, the curriculum should not be separated from socio-cultural influence. Not taking the context (e.g., culture) into account, the analysis, specifically cross-national comparison, might tell only a partial story or even reach misleading conclusions. All the factors described above point to some possible areas for future investigation in the context of comparative textbook analyses. Furthermore, the question of whether learning more content knowledge at earlier grades can lead to better knowledge development for students at different achievement levels remains unanswered.

Looking ahead, we plan to pursue several different directions. The first one is to expand our textbook analysis of the concepts of triangle congruence to include the analysis of its connection to other curriculum units. Such analysis can then be connected back to the curriculum standards to provide a more complete picture of the intended learning trajectories of this important geometric concept. The second direction is to study the enactment of triangle congruence lessons in the U.S. and Chinese classrooms. Specifically, we would like to see how the teachers and students jointly tell the story of the congruent triangles. We also would like to investigate the impact of using different types of tasks on student learning that will answer our own questions: "What learning opportunities can real-life problems and decontextualized problems provide?" and "What obstacles would students face encountering these different contexts?" In addition, we plan to consider examining assessments regarding the topic from the two countries and discuss the relationship as well as the alignment between the intended, enacted, and assessed curriculum.

In this paper, we shared lessons we have learned from our own curriculum studies. Specifically, we used examples to illustrate adapting theoretical frameworks and using comparative methods in analysis. Hopefully, these lessons can provide a reference for future researchers who conduct curriculum studies.

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