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The Impact of the Pirie-Kieren Theory on Developing Fraction Understanding in Third-Grade Students

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ABSTRACT
This study aims to examine the impact of applying the Pirie-Kieren theory to enhance third-grade students' understanding of fractions in Kosovo. Fractions are a foundational concept in mathematics, essential for both mathematical competency and general education. However, comprehending fractions remains a major challenge for students, educators, and parents worldwide, including in countries with well-established traditions in mathematics education. By incorporating the Pirie-Kieren theory into the design of instructional programs and textbooks, the content is structured in a coherent and logical progression. This method ensures that each learning unit builds on previously acquired knowledge, thereby transforming present learning into a strong foundation for future understanding. In Kosovo, where the introduction of fractions begins in the third grade, our study utilized an experimental design with a control group, involving a total of 148 students - 80 from experimental classes and 68 from control classes - across three schools in different regions. We started by conducting a detailed analysis of curriculum components and textbooks related to fractions, identifying several didactic-methodological shortcomings, including disorganized content, inadequate visual representations, and material that was not age-appropriate. In the experimental phase, we created detailed lesson plans for teachers and worksheets for students in the experimental groups, ensuring they were in accordance with the principles of the Pirie-Kieren theory. These materials were refined with input from experts and reputable organizations such as UNESCO, the State Department, NCTM, and OECD. The study also designed a specialized test to evaluate outcomes in both the experimental and control groups. The results were analyzed using various statistical methods to compare the performance of the two groups. The study findings reveal that teaching and learning fraction-related content through the Pirie-Kieren theory/model is significantly more effective than traditional instructional methods.

KEYWORDS
Fractions; fraction understanding; Pirie-Kieren theory; students.

INTRODUCTION

Fractions are a fundamental concept in mathematics, playing a central role in mathematical education. Fractions are especially important as they require a more profound understanding than whole numbers. They also play a vital role in general education. However, despite this importance, many children and adults still struggle with grasping the properties and magnitudes of fractions, even after years of formal education. An inadequate understanding of fractions in elementary school often results in poor mathematical performance in algebra and impedes overall academic achievement in high school. Teachers recognize the importance of fractions, viewing students' proficiency in this area as either a major advantage or a substantial barrier to their academic success (Siegler et al., 2012).

This research is part of the author's doctoral dissertation, focusing on the implementation of the Pirie-Kieren theory to enhance fraction comprehension in primary education. There is a common consensus among educators and researchers that, before students can effectively learn and understand fractions, they must possess a solid foundation in natural and whole numbers, as well as a clear understanding of division and the partitioning of whole quantities into equal parts. Robert Siegler emphasizes the importance of building on students' prior knowledge to clarify the meaning of fractions.

Children often arrive at school with a foundational understanding of fractions. For instance, if asked to divide something equally, they can accomplish this by alternately counting items for each person, such as one for you, one for me, and so forth (Siegler, 2010, p. 2).This study uses a quantitative, experimental design that includes a control group. The research took place in three schools: Primary School "Pavarësia" in Prishtina, Primary School "Bafti Haxhiu" in Viti, and Primary School "Dëshmorët e Vitisë" in Viti, with a total of 148 third-grade students participating. In each school, one class was chosen as the experimental group, while another served as the control group. All instructional materials for teachers and students in the experimental classes were pre-developed based on the principles of the Pirie-Kieren theory. In the experimental classes, both teachers and students used materials specifically designed by the researcher to teach fractions based on the Pirie-Kieren theory. Subsequently, both the experimental and control groups completed a standardized test to assess the effectiveness of the Pirie-Kieren theory in improving fraction comprehension. The data obtained from this research underwent a series of analyses and statistical tests, revealing substantial findings that highlight the benefits of applying the Pirie-Kieren theory to fraction comprehension. I anticipate that these positive results will stimulate further research on the application of the Pirie-Kieren theory to other mathematical concepts, ultimately encouraging its broader adoption in mathematics education in Kosovo, similar to its implementation in advanced educational systems globally.

LITERATURE REVIEW

The understanding of fractions is not only essential with in the field of mathematics but also extends to other disciplines and practical activities. Fractions are essential for developing a competent individual capable of functioning effectively in various workplace settings. Conceptual recognition and understanding are necessary for operations with fractions, which often contradict the principles children have learned about natural and whole numbers. Fractions are a mathematical concept that students will encounter throughout their education and beyond (NCTM, 2021).

Numerous studies on the difficulties of learning and understanding fractions have been carried out by specialists and a variety of national and international institutions, such as UNESCO, OECD, the U.S. Department of State, along with other governmental and non-governmental organizations. A study titled "The Importance of Fractions Instruction" by the Institute of Education Sciences (IES) identifies several misconceptions and difficulties that students encounter in mastering fractions. Several students struggle with fraction arithmetic, finding challenges in all four operations. They often confuse fractional operations with those of natural and whole numbers, leading to errors. For example, students frequently make mistakes in calculating a fraction of a number, determining which fraction is larger or smaller, and adding fractions by incorrectly applying the rules of natural numbers.

The major misconception children have with fractions is that they treat them like whole numbers. For example, when solving a problem like $\frac{3}{5}+\frac{5}{6}$, they might give an answer like $\frac{8}{11}$ incorrectly adding the numerators and denominators as if they were whole numbers. Children struggle to learn the procedures because they lack a basic conceptual understanding of what fractions represent. Without a clear sense of the relative sizes of $\frac{12}{13}$ is and $\frac{7}{8}$ they are just as likely to say that 12/13 + 7/8 = 19 or 21, they are to estimate that the sum is about 2, since $\frac{12}{13}\approx 1$ and $\frac{7}{8}\approx 1$ (Siegler, 2010).

Research indicates that many students incorrectly assume that multiplying fractions will always yield a larger fraction, while dividing fractions will always result in a smaller one. They also incorrectly assume that dividing by zero yields a number. These misconceptions stem from a flawed understanding of fractions based on oversimplified mental schemas for number operations that fail to account for the complexity of fractions (Ciosek & Samborska, 2016).

Many of these studies highlight the need for improved teacher training in mathematics, particularly in the area of fractions. An important number of teachers possess only a superficial understanding of fractions. While they may know the procedures for working with fractions, they often struggle to explain and justify these procedures to students. Without a deep comprehension, teachers are unable to effectively demonstrate the underlying logic of mathematical rules, causing students to rely on rote memorization rather than genuine understanding-knowledge that is likely to fade over time. From this body of research, it is evident that the key challenges lie in determining when and how fractions should be taught. Regarding the timing of instruction, researchers offer two primary recommendations: (1) In countries with well-developed educational systems, fractions should be introduced in the early grades, such as second or third grade (Fazio & Siegler, 2011); (2) In countries with struggling educational systems, it is recommended to introduce fractions starting in the fifth grade, with expanded instruction in the seventh grade (Brdar et al., 2014).

Kosovo, as a newly established state, has its pre-university education system guided by a curriculum that emphasizes constructivist principles, including the Pirie-Kieren theory. In Kosovo, students are introduced to fractions in the third grade, making it an ideal context for researching fraction comprehension through the framework of the Pirie-Kieren theory. The Pirie-Kieren theory, recognized as an effective framework for developing and deepening mathematical understanding, is widely implemented in countries with strong traditions in mathematics education, particularly in the design of curricula and textbooks. The theory examines the dynamics of mathematical understanding, describing it as a dynamic, non-linear process. The model outlines eight unique levels of understanding, starting from the basic "primitive knowing" and progressing to the advanced level of "inventising," which represents the evolution of an individual's understanding of a particular mathematical concept (Borgen, 2006).

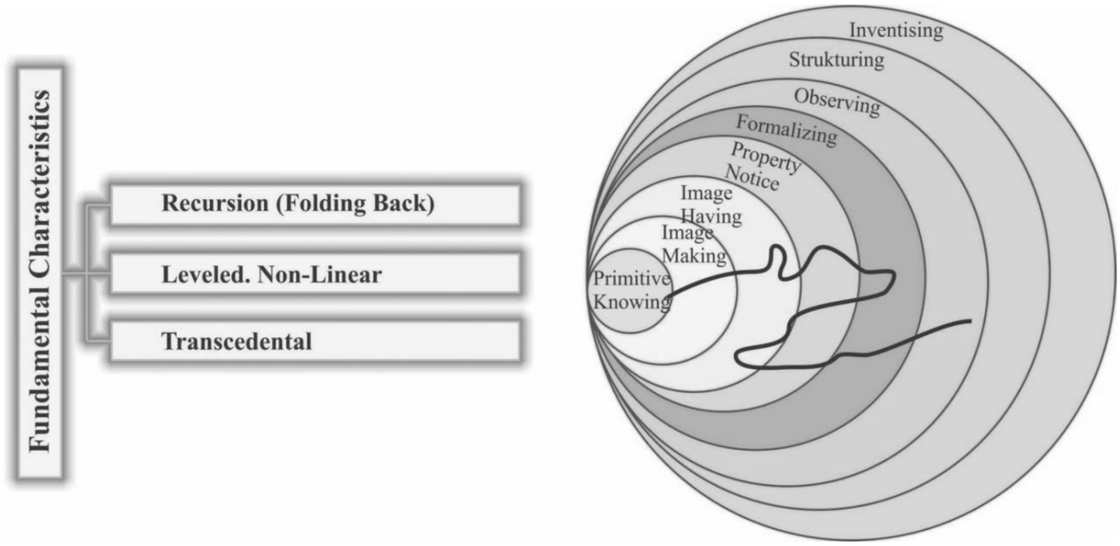
Rooted in constructivist principles, the Pirie-Kieren theory was developed by Dr. Susan Pirie and Dr. Thomas Kieren, both prominent figures in mathematics education research. Despite working in different locations—Pirie at the University of Oxford and Kieren at the University of Alberta—their collaboration began in 1988, driven by a shared belief in the dynamic nature of learning. They posited that understanding is not a static state but a continuous process of development (Borgen, 2006).

The original concept of mathematical understanding in the Pirie-Kieren theory is rooted in von Glasersfeld's (1987) constructivist framework, which defines understanding as a continuous process of organizing knowledge structures. According to the Pirie-Kieren theory, the development of mathematical understanding is seen not as a straightforward progression but as a dynamic and recursive process, where understanding grows through interactions with materials, peers, and teachers in various contexts (Mokwebu, 2013).

This theory closely aligns with the numerous recommendations from scholars, methodologists, and educational institutions, both nationally and internationally, which support teaching fractions using models and real-world problems, with a focus on fostering quantitative understanding. The deliberate use of the term "image" in this model reflects the researchers' belief that these levels of understanding are grounded in both visual representations and mental imagery (Mokwebu, 2013).

Figure 1.

Characteristics of the Pirie-Kieren theory. (adapted from Martínez, 2017, slide 25)



The Pirie-Kieren model is defined by its recursive nature, especially through the concept of "folding back." When students face a problem they cannot solve, they are encouraged to revisit earlier, more fundamental levels of understanding. This process is essential for expanding comprehension and underscores the non-linear nature of mathematical understanding (Iwata &Yasunaga, 2016). While this action might at first appear to be a step backward in the learner's observable behavior, it frequently results in a more profound understanding of the mathematical concept, representing a crucial phase in the dynamic development of mathematical knowledge (Martin et al., 2005).

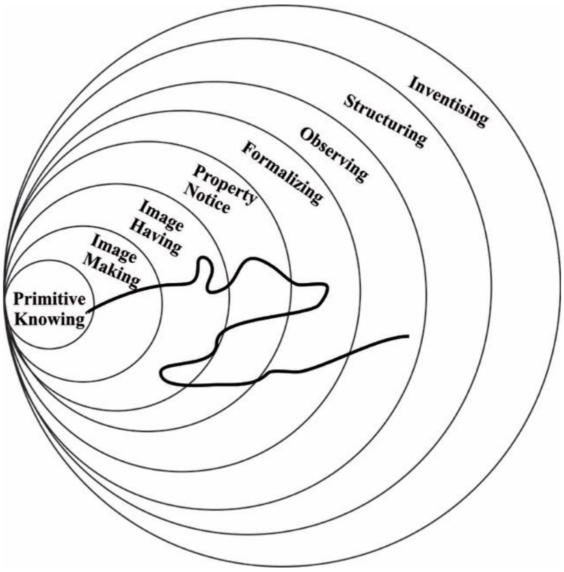
Another notable feature of the Pirie-Kieren model is the shift between action and expression. This is evident in the arbitrary transitions between levels, as learners move back and forth to meet specific demands or expand their understanding. In essence, Pirie and Kieren showed that the growth of understanding is not linear but involves a continuous interplay of actions across different levels. Each level builds upon prior understanding, ensuring continuity and interconnection with the internal levels (Iwata &Yasunaga, 2016).

The eight levels of understanding described in the Pirie-Kieren theory can be applied to analyze the progression of mathematical comprehension in individuals, such as students or professionals, as well as within particular content areas. His framework is particularly valuable within defined contexts or over extended periods, such as a learning unit, academic term, age group, or class. The application of these levels is determined by the researcher and shaped by the specific focus of their study (Martin et al., 2005).

Pirie and Kieren consider mathematical understanding to be a multifaceted process that cannot be neatly divided into a few distinct categories or simplified to a single acquisition. They portray mathematical understanding as a leveled but non-linear, recursive process, where a student's thinking shifts between different levels of comprehension. Each level is interconnected, with the next building upon the previous ones (Pirie & Kieren, 1994). Their theory frames understanding as a personal and reorganized construction of knowledge structures, applicable to both individuals and groups.

Figure 2.

Pirie-Kieren theory. (Adapted from Pirie-Kieren, 1994)



The first level, "primitive knowing," does not suggest a rudimentary level of mathematical knowledge; instead, it marks the stage where mathematical understanding starts to grow. This level encompasses all the knowledge a student brings when tackling a new mathematical task or learning a new concept. "Primitive knowing" serves as the foundation for all further understandings and is incorporated into subsequent levels. The term "primitive" is used to mean "primary" or "basic," without implying anything about the complexity of the mathematical knowledge involved. It includes everything a person has "in mind" about the current task, excluding their understanding of the specific subject (Gibbons, 2014).

At the second level, "image making," students start to recognize and utilize their primitive knowledge in novel ways. This stage is essential and dynamic in the development of understanding. Through different activities, students create images like pictures, graphs, language-based representations, actions, or numbers, drawing on their existing knowledge (Iwata &Yasunaga, 2016).

The third level, "image having," represents the shift from "image making" to the acquisition of mental images, a crucial step in a student's mathematical understanding. At this stage, students often feel they have fully grasped the concept. However, it's important to recognize that while the student may have formed a conceptual image, it could still be incomplete or inaccurate, which may lead to challenges and obstacles in further developing their understanding of the topic (Borgen, 2006).At the fourth level, "property noticing," students start to manipulate and combine aspects of the images they created in earlier stages to establish specific contexts and recognize essential features of a given topic. This stage involves identifying differences, combinations, or connections between images and understanding the relationships among these elements. The key distinction between "image having" and "property noticing" is the student's capacity to observe and articulate the relationships between images and to verify these connections.

At the fifth level, "formalizing," students extract common features from previous images to form formal mathematical ideas. This involves consciously reflecting on the properties they've observed and recognizing shared characteristics. At this stage, students generalize from specific images and begin to think in broader terms. The properties and generalizations identified here lead to the development of mathematical definitions. While the language used doesn't need to be formal mathematical terminology, the students' descriptions should align with the correct mathematical definitions (Gibbons, 2012). Generalization occurs at this level, as students realize that a particular method applies to all cases within a category. Through abstraction, they begin to engage in more formal mathematical reasoning, moving beyond specific instances to work with broader concepts (Borgen, 2006). The sixth level, "observing," involves a deliberate effort by the individual to deepen their understanding of the concept at hand. Having already developed a formalized understanding of various mathematical aspects, the student actively searches for patterns or relationships among them, seeking commonalities to grasp the "big picture" (Borgen, 2006). At the "observing" level, the student reflects on the formal activities they engaged in during the formalizing stage and seeks to identify patterns and connections. This process helps them define their formal ideas as algorithms or theorems. At this stage, the student is able to organize their ideas coherently and logically, creating theories based on these ideas. Once these theories are developed through observation, the natural expectation is to determine their validity.

At the seventh level, "structuring," the student uses reasoning and evidence to justify or prove their mathematical theories. For a deeper understanding, the student must be able to explain why their formal observations are likely true or consider why they might not hold true. At the eighth and final level, "inventising", the student moves beyond the structural understanding they have previously constructed, asking new questions that lead to the development of new and different meanings or concepts. At this level, mathematical understanding becomes limitless, transcending the existing structure and allowing the student to explore questions of the form "what if?"

RESEARCH METHODOLOGY

Purpose and Objectives of the Study

The primary objective of this study is to examine the effects of applying the Pirie-Kieren theory on the development and enhancement of third-grade students' understanding of fractions in Kosovo. By comparing the impact of this theoretical approach with traditional teaching methods, the study aims to assess its effectiveness in improving students' comprehension of fractions.

Sample Selection

The study sample consists of 148 third-grade students from three primary schools in Kosovo: "Pavarësia" School in Prishtina, “Bafti Haxhiu” School in Viti, and "Dëshmorët e Vitisë" School in Viti. The selection of students was intentional, aimed at ensuring diverse representation from both urban and rural areas. Two third-grade classes from each school were chosen to participate in the study, with one serving as the experimental group and the other as the control group.

Data Collection

Data for this study were collected from two primary sources:

- **Analysis of Curricula and Textbooks:** A thorough analysis of the third-grade mathematics curricula and textbooks was conducted to evaluate their suitability for the students' age and the extent to which they incorporate feedback and recommendations from relevant experts and institutions regarding the challenges of understanding fractions.
- **Written Tests:** Written assessments were administered to evaluate students' knowledge of fractions both before and after the intervention. These tests were designed to measure knowledge acquisition in both the experimental and control groups.

Research Variables

Independent Variables:

- Active Variables: These include the Pirie-Kieren theory, along with the specially prepared instructional materials and student worksheets designed according to this theory.
- Attribute Variables: These refer to inherent characteristics of the population, such as the students' grade level (third grade) and the location of the schools (urban vs. rural).

Dependent Variable: The dependent variable in this study is the students' understanding of fractions, measured using a dichotomous categorization (correct/incorrect).

Implementation of the Study

In the experimental classes, the teaching and learning of fractions were conducted using materials and worksheets specifically designed according to the Pirie-Kieren theory. In the control classes, instruction followed standard textbooks and regular lesson plans. To ensure the proper implementation of the methodology, special meetings were held with the teachers involved in the study to explain the research methodology and objectives. It was essential that the teachers fully understood the application of the Pirie-Kieren theory and the use of the prepared materials.

Data Analysis

After the tests were completed, the data were input into a computerized system for processing. The analysis involved coding the tests and responses, filtering the data, and conducting a t-test to identify any statistically significant differences between the experimental and control groups. The t-test was applied to compare the mean scores of the two independent groups (experimental and control classes), assuming equal variances.

Research Methods

This study employed both descriptive and experimental research methods:

- **Descriptive Method:** Utilized for the collection, processing, and interpretation of data.
- **Experimental Method:** Applied to compare the effects of instruction using the Pirie-Kieren model with traditional teaching methods.

The research was conducted through the following steps:

- **Design and Preparation of Tests:** The tests were designed in alignment with the Pirie-Kieren theory.
- **Implementation of Tests:** The testing process was supervised to ensure consistency and accuracy.
- **Data Control and Processing:** The collected data were meticulously controlled and processed to ensure their validity and reliability.

In conclusion, the methodology employed in this study rigorously evaluates the effectiveness of the Pirie-Kieren theory in enhancing third-grade students' understanding of fractions, while also comparing this approach with traditional teaching methods.

FINDINGS/RESULTS

Fractions in Teaching Programs for Grade III

The mathematics teaching programs in Kosovo are highly generalized, focusing primarily on broad learning outcomes that emphasize overarching mathematical competencies. Despite attempts to reform the educational program between 2011 and 2014, these efforts ultimately stalled. It seems that educational authorities in Kosovo were satisfied with producing general foundational documents for various education levels and subject areas, without addressing the need for targeted documents and strategies to support effective teaching in specific fields, subjects, and concepts. These shortcomings are particularly evident in the instruction of fractions, a topic often considered challenging.

In Kosovo, fractions are introduced in the third grade. The Kosovar mathematics curriculum outlines that third-grade students are introduced to fundamental fraction concepts, such as equivalent fractions, comparing fractions with the same numerator or denominator, and adding and subtracting fractions with a common denominator (MASHT, 2019). The mathematics teaching program provides only a brief and generalized list of learning outcomes for fractions, despite the fact that they are being introduced to children for the first time. As a result, there is a need for a more extensive and detailed list of learning outcomes to guide textbook authors and teachers in developing effective teaching content for this topic. Specifically, additional outcomes and greater emphasis are required to address the foundational understanding of fractions and to clarify the meaning of their elements, such as denominators and numerators.

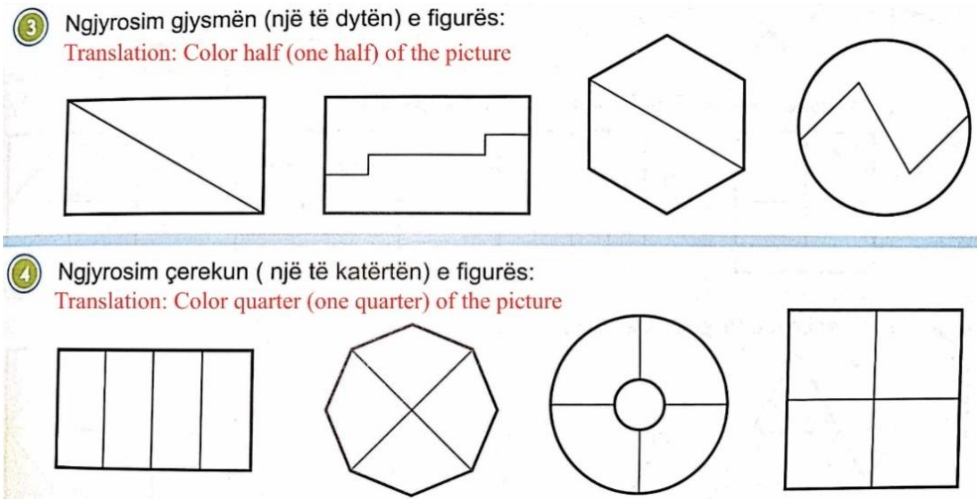
Due to the limited number of learning outcomes, the existing ones are presented in a non-logical sequence. For instance, how can a student, especially a third-grader, "distinguish fractions that represent the same part of a whole" (outcome 2) before they have learned to "determine the part of the content" (outcome 3)?

Fractions in Textbooks for Grade III

In Kosovo, the following textbooks are used for teaching mathematics in grade III: "Mathematics 3" and "Mathematics 3 – Workbook". Both textbooks, along with all their content, were authored by Ramadan Zejnullahu and Sejdi Bilalli. In the main textbook for grade III, fractions are covered from pages 151 to 156, and in the workbook from pages 92 to 94. Both texts are from the 2005 edition. Due to space limitations, we will focus on some of the most important and representative examples related to fractions in the main mathematics textbook for grade III. From the outset (p. 151), when fractions are introduced, their irregular presentation becomes evident. Rather than gradually and carefully guiding students through the concept of fractions, the material is presented abruptly, despite this being the students' first encounter with the topic. These texts include examples of representational models that could easily confuse students, such as examples 3 and 4 (p. 151).

Figure 3.

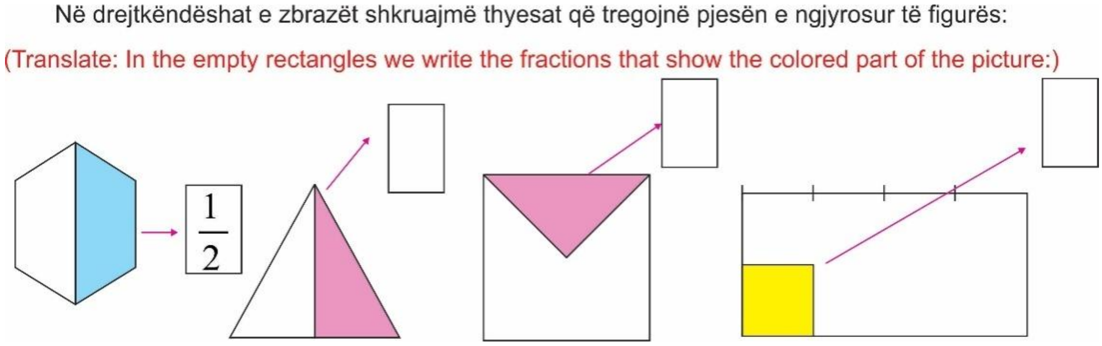
Models not suitable for the representation of fractions



In example 3, the figures are divided into two equal parts, but in a way that does not align with the cognitive abilities of students at this grade level. Students of this age may find it difficult to recognize that the figures are evenly divided. The prior recommendation—that figures (representative models) should be divided vertically when illustrating equal parts—has not been followed. In example 4, which asks students to color one-quarter (1/4) of a figure, the third figure is especially confusing. The small circle in the center creates ambiguity for the student, causing uncertainty about how to handle it. Is the circle considered part of the figure or not? Despite both textbooks being authored by the same individuals, there is no logical connection between them, and they do not function cohesively as a pair. In the Workbook, fractions are covered in just three pages under the single title Initial Knowledge of Fractions, without clearly differentiating the content, instructions, and tasks according to the relevant topics outlined in the teaching program.

Figure 4.

Models not suitable for the representation of fractions



Analysis of Lesson Plans for Teachers and Worksheets for Students in the Experimental Groups

Groups In response to the identified gaps and deficiencies in existing programs and textbooks, we created lesson plans for teachers and worksheets for students, tailored specifically for the experimental classes. These materials were designed following the principles of the Pirie-Kieren theory and the recommendations of experts and relevant institutions. Additionally, they align with the learning outcomes outlined in the grade III mathematics curriculum.

Analysis of Test Results

In the analysis, the following variables will be considered:

- **Class:** 3rd grade, coded as 3.
- **Groups (Classes):** The experimental and control groups are coded as follows: 1 represents the experimental group (class), and 2 represents the control group (class).
- **Experimental Variable:** The application of the Pirie-Kieren theory.

Table 1.

General Statistics

General Statistics					
	(1) experimental (2) control	N	Mean	Std. Deviation	Std. Error Mean
Res-test	1	80	9.84	1.739	.194
	2	68	7.32	3.150	.382

Table 1 shows substantial statistical differences between the experimental and control groups. Specifically, the experimental group achieved a higher average number of correct answers (M=9.84) compared to the control group (M=7.32). Furthermore, the standard deviation is notably lower in the experimental group (SD=1.74) than in the control group

(SD=3.15), indicating that the results within the experimental group are more consistent and homogeneous than those in the control group.

Table 2 (see appendix)

Table 2 indicates a statistically significant difference between the experimental and control groups for the third grade ($t = 5.865$, $df = 100.485$, $p<0.01$). This confirms the presence of a statistical difference between the groups, which can be attributed to the implementation of the Pirie-Kieren theory. Thus, we have evidence that the Pirie-Kieren model led to significantly greater success for third-grade students compared to traditional teaching methods.

Analysis of the Impact of Applying the Pirie-Kieren Theory on Attaining the Learning Outcomes Targeted in the Study

By developing teacher lesson plans and student worksheets for the experimental classes, based on the Pirie-Kieren theory, our goal was to design content that is both suitable and effective in achieving the learning outcomes for fractions, as outlined in the third-grade curriculum provided by the Ministry of Education, Science, and Technology (MEST).Below, we outline the specific fraction-related learning outcomes related to fractions for third grade, which served as the focus of our research.

Student Outcomes:

- Identifies parts of a picture represented by fractions.
- Provides written justification for why a fraction represents the colored parts of a picture.
- Determines which of the given fractions correspond to the colored parts of a rectangle.
- Identifies the fractions that represent the colored parts of a circle. Colors the number of parts in figures divided into equal sections according to the corresponding fractions.
- Associates various geometric representations with their corresponding fractions.
- Writes the fraction representing the number of distinct parts as on the number line.
- Recognizes the parts of a number line as represented by fractions.
- Identifies and connects equivalent fractions (fractions that represent the same quantity).
- Compares fractions that have the same denominator.
- Compares fractions with the same numerator.
- Calculates fractional parts of a whole.

To evaluate the extent to which these outcomes were achieved in both the experimental and control groups, a corresponding assessment was developed. Each outcome was converted into a specific task or question on the test. The test comprised seven tasks with a total of 12 requirements. The results are presented in Table 3 and Figure 5.

Table 3 (see appendix)

Figure 5 (see appendix)

Table 3 and Figure 5 clearly demonstrate a substantial difference in the achievement of most learning outcomes, with the experimental group outperforming the control group among third-grade students. This difference can be attributed to the implementation of the Pirie-Kieren theory. The experimental group achieved an average of 81.14% correct answers, compared to 61.19% in the control group, indicating a substantial difference. For each learning outcome, students in the experimental group consistently outperformed those in the control group, demonstrating higher success in achieving the targeted objectives.

Overall, it can be concluded that the application of the Pirie-Kieren theory has been highly effective in achieving the intended outcomes among the third-grade students participating in the study.

DISCUSSION OF RESULTS

The findings of this study reveal that students in the experimental group, taught using the Pirie-Kieren theory, exhibited substantial improvements in their understanding of fractions compared to the control group. Specifically, students in the experimental group showed a marked increase in their ability to identify equivalent fractions and perform basic operations with them. For example, in the task requiring students to identify parts of a picture representing fractions, 91.36% of students in the experimental group successfully completed this task, compared to only 53.73% in the control group. Additionally, 71.6% of students in the experimental group were able to provide written justifications explaining why a fraction represented the colored parts of a picture, in contrast to 41.79% in the control group. Similarly, in tasks involving fraction comparison, students in the experimental group achieved higher percentages of correct responses—81.48% for comparing fractions with the same denominator and 77.78% for comparing fractions with the same numerator—compared to the control group, which scored 56.72% and 49.25%, respectively. These significant differences highlight the marked positive impact of the Pirie-Kieren theory on students' comprehension of fractions. Additionally, the experimental group surpassed the control group in tasks that involved identifying fractions representing the colored sections of geometric shapes, such as rectangles and circles. For instance, in one task, 95.06% of students in the experimental group correctly identified the fractions for a circle, compared to 71.64% in the control group. In another similar task, 97.53% of the experimental group succeeded, compared to 85.07% of the control group. The results of this study offer compelling evidence for the effectiveness of the Pirie-Kieren theory in improving students' comprehension of fractions. The theory's structured approach, which builds on prior knowledge and fosters deep conceptual understanding, has been shown to be significantly more effective than traditional teaching methods. These results not only demonstrate the educational benefits of the theory but also carry important implications for curriculum development. By adopting this approach, teachers can substantially improve students' comprehension of fractions, signaling a need for curriculum revision and the creation of more suitable teaching materials in Kosovo. Moreover, these findings may encourage policymakers to incorporate innovative methodologies like the Pirie-Kieren theory into the education system, potentially serving as a model for future educational reforms.

Limitations of the Study

While this study offers valuable insights into the impact of applying the Pirie-Kieren theory to enhance third-grade students' understanding of fractions, several limitations should be acknowledged:

- Limited Sample and Representation: The study involved students from only three schools in Kosovo, selected through purposive sampling, limiting the generalizability of the results to other populations and educational contexts.
- Variability in Implementation: Differences in how teachers implemented lessons based on the Pirie-Kieren theory may have influenced outcomes in both the experimental and control groups.
- Limitations in Testing Knowledge: External factors, such as test anxiety and variations in question quality, may have affected the accuracy of test results, potentially skewing measurements of students' knowledge acquisition.
- Subjectivity in Curriculum and Textbook Analysis: The analysis of curricula and textbooks was conducted from a specific perspective, and differences in their implementation across schools may have impacted the results.

Short-Term Focus: The study examined only the short-term effects of the intervention, without assessing its long-term impact on students' acquisition of mathematical concepts. Despite these limitations; the research provides valuable insights into the application of the Pirie-Kieren theory in primary education. It also suggests that future studies could address these limitations to obtain more generalizable and in-depth results.

CONCLUSIONS

- In Kosovo, there is a noticeable lack of comprehensive research on the teaching and learning of fractions, leading to ad-hoc approaches in educational programs and textbooks that negatively impact both the structure and scope of the content being taught.
- The teaching programs and textbooks that address fractions suffer from methodological and didactic shortcomings, including an illogical arrangement of content, insufficient visual aids, and a lack of alignment with students' age or grade level.
- These texts also lack a robust didactic apparatus—supplementary components that support and enhance the basic text, such as questions, tasks, expressions, vocabulary, illustrations, comments, and instructions, which are essential for deeper understanding.

Recommendations

- Given the critical importance of fractions in mathematics and other fields, Kosovo urgently needs comprehensive research on the teaching and learning of fractions and other mathematical concepts. Such research should focus on identifying the most appropriate age or grade for introducing fractions and determining the most effective methods and strategies for teaching them to students.

• The possibility of adapting mathematics textbooks, especially those covering fractions, from countries with more advanced educational systems should be explored. However, careful adaptation is essential to ensure that the content aligns with the specific context and realities of Kosovo.

• Based on the study findings, immediate revisions should be made to teaching programs, textbooks, and instructional methods. In many countries, research has led to the development of guidelines and strategies for teaching fractions.

• Kosovo should leverage support from UNESCO, which developed a strategy for textbooks and teaching tools in 2005, as well as other relevant institutions. These organizations have shown a willingness to assist in building a robust community of textbook authors and evaluators, encouraging the involvement of a wide range of teachers and experts. Such collaboration would help create simpler, more engaging texts for students (Noti, 2013).

Given the strong statistical evidence demonstrating the success of the Pirie-Kieren theory in teaching fractions to third-grade students, I propose the following recommendations:

• Educational authorities in Kosovo should explore opportunities to test this theory with other mathematical topics various age groups or grades.

• The Pirie-Kieren theory should be incorporated when designing curricula and textbooks, especially for mathematical topics like fractions, due to its leveled yet nonlinear, recursive, and transcendental approach. Its application would create more cohesive and interconnected curricula, textbooks, and specific mathematical content The application of the Pirie-Kieren theory would also streamline and facilitate the assessment of students' knowledge. Its structured approach to content leveling and ranking facilitates the efficient and straightforward measurement of achievement.

• Researchers and evaluators of mathematics curricula and textbooks could adopt the Pirie-Kieren theory as an effective framework or lens for evaluating content and instructional design.

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APPENDIX

Table 2.

Data from the t-test

Independent Samples Test										
		Levene's Test for Equality of Variances		t-test for Equality of Means						
		F	Sig.	T	Df	Sig. (2-tailed)	Mean Difference	Std. Error Difference	95% Confidence Interval of the Difference	
									Lower	Upper
Res-test	Equal variances assumed	27.473	<.001	6.126	146	<.001	2.514	.410	1.703	3.325
	Equal variances not assumed			5.865	100.485	<.001	2.514	.429	1.664	3.364

Table 3.
The percentage of achieving the learning outcomes

Intended learning out comes/tasks	Percentages of achievement of learning out comes for experimental and control groups (classes)	
	Experimental	Control
Identifies parts of the picture how much tell fractions	91.36	53.73
Justifies in writing why the fractions hows the colored parts of the picture	71.6	41.79
Determines which of the given fractions present the colored parts of the rectangle	88.89	70.15
Determines which of the given fractions present the colored parts of the circle	95.06	71.64
Determines which of the given fractions present the colored parts of the circle	97.53	85.07
Itconnectsthedifferentgeometricrepresentationswiththecorrespondingfractions	93.83	77.61
Write the fraction that shows as many distinct parts as there are on the number line	56.79	56.72
Distinguishes as many parts of the numeral line as fraction show	80.25	65.67
Interrelates equal fractions among themselves (fractions showing the same size)	71.6	50.75
Compares fractions with the same denominator	81.48	56.72
Compares fractions with the same numerator	77.78	49.25
Calculate parts of a whole	77.78	47.76

Figure 5.

Graph of the degree of achievement of learning out comes for the experimental group and the control group