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Transformation of Didactic–Mathematical Knowledge and Competence Teacher (DMKCT): Analysis of Elementary School Students' Mathematical Way of Thinking (WoT)

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ABSTRACT
Although mathematics teaching has been widely studied, limited attention has been given to teachers' Didactic-Mathematical Knowledge and Competence (DMKCT) and students' Ways of Thinking (WoT). This study investigates the transformation of teachers' knowledge and didactical competence in teaching mathematical concepts, as well as students' WoT when receiving mathematical information from teachers and solving problems. The study employed Didactical Design Research (DDR) developed by Suryadi, drawing on interpretive and critical paradigms. Data were collected through observations of mathematics learning, written tests, and in-depth interviews. The research was conducted in three stages: prospective analysis, metapedadidactic analysis, and retrospective analysis, which led to a revised didactical design. The participants were one fifth-grade teacher and the students of the teacher's class at a public elementary school in East Java, Indonesia. The findings revealed two main aspects of DMKCT. The first is the epistemic aspect, in which teachers explained basic mathematical concepts related to fractions. In this aspect, students' WoT emerged as rule-based (formal) thinking, but their ability to represent and visualize mathematical objects remained limited. This indicated that the Concrete–Representational–Abstract (CRA) sequence had not been fully realized in students' mathematical thinking. The second is the interactional aspect, in which teachers did not explain mathematical concepts in sufficient detail and provided limited interaction with students, resulting in teacher-centered learning. Therefore, students experienced misconceptions in understanding fractions and completing the test. Students' WoT did not appear clearly because of their limited opportunities to process and construct information during learning. These misconceptions indicate the need for appropriate scaffolding to support students' understanding and prevent difficulties in learning more complex mathematical concepts.

KEYWORDS
Didactic-Mathematical Knowledge and Competence; Didactical Design Research; Mathematics learning; Students' Ways of Thinking.

INTRODUCTION

Didactic-Mathematical Knowledge and Competence for Teachers (DMKCT) has recently gained increasing attention because it is essential in mathematics teaching. Herrera-García and Dávila-Araiza (2020) stated that teachers need didactic-mathematical knowledge of various aspects or dimensions involved in the learning process, including epistemic, ecological, cognitive, affective, mediational, and interactional dimensions. Similarly, Pino-Fan et al. (2011) explained that the Didactic-Mathematical Knowledge model interprets and characterizes teachers' knowledge through three broad dimensions: mathematical, didactical, and meta-didactical-mathematical knowledge.

Previous studies have shown that didactic-mathematical knowledge can be analyzed through teachers' ability to construct and convey mathematical arguments (Ayalon &Hershkowitz, 2018; Castro et al., 2021; Metaxas et al., 2016; Nielsen, 2011). Furthermore, students' perceptual ability to understand information from teachers can be psychologically analyzed (Braun et al., 2006; Fernando et al., 2017; Medsen et al., 2024). However, previous studies have not sufficiently connected teachers' didactical abilities in mathematics teaching with students' cognitive activities. Students' ability to receive and process information from teachers is strongly influenced by their cognitive activities during mathematical thinking processes. Therefore, this study aims to explore students' Ways of Thinking (WoT) in receiving mathematical concepts based on teachers' Didactic-Mathematical Knowledge and Competence.

The current study focuses on two main aspects: teachers' didactic-mathematical competence and students' cognitive based on WoT. Teachers' didactic-mathematical competence is examined through their ability to convey mathematical concepts to students. Therefore, Didactical Design Research (DDR) is used to analyze didactical situations and to develop didactical designs for mathematics instruction, as well as to identify students' WoT. DDR has recently been applied to fraction learning to identify ontogenic, didactical, and epistemological obstacles and to revise a didactic design after implementation (Isnawan et al., 2026). Gravemeijer and Cobb (2006) explained that design research is grounded in the effort to create educational innovations. In line with this, Suryadi (2019a) stated that DDR is based on two paradigms, namely interpretive and critical paradigms. These paradigms serve as the foundation for developing effective didactical designs that can be applied in learning activities. In this study, DDR refers to the Didactical Design Research developed by Suryadi (2013, 2019a), which is rooted in Brousseau's (1997) theory of didactical situations. It is different from Design and Development Research (Richey & Klein, 2007), which focuses on developing and evaluating instructional products or models.

Cognitive is closely related to students' WoT when understanding the mathematical concepts conveyed by the teacher and when completing tests. The analysis of teachers' didactical competence in this study adopts Castro's (2021) framework and is combined with Valsiner's theory. Meanwhile, students' WoT is identified based on their mathematical thinking ability. Mathematical thinking does not merely refer to thinking about mathematics as a subject,

but also involves dynamic cognitive processes (Zubeyde, 2023). Mathematical thinking as a combination of estimation, induction, generalization, decision making, deduction, description, verification, and formal and informal reasoning (Liu, 2003).

Mathematical thinking ability is an essential ability that needs to be possessed by students (Bhagwonparsadh & Pule, 2024; Faizah et al., 2020; Stacey, 2006), because it plays a role in forming logical, critical, and systematic thinking patterns (Purwasih et al., 2024; Soboleva et al., 2020). In addition, mathematical thinking supports students in making data-based decisions, improving problem-solving skills, and developing perseverance and rigor when facing challenges (English & Gainsburg, 2015; Novikasari et al., 2024; Özdemir & Bostan, 2021). It also serves as an important foundation for the development of science and technology (Ernest, 2002; Sternberg, 2012).

In this study, mathematical thinking is understood as a higher-order thinking process required in problem-solving situations. It involves estimation, induction, generalization, sampling, deduction, description, verification, formal and informal reasoning. Developing mathematical thinking from an early age is a strategic effort to prepare students to become intelligent, adaptive, and innovative learners. Therefore, this study seeks to address the gap in the literature by exploring the transformation of mathematics instruction through DMKCT and students' WoT.

This study addresses the existing research gap by integrating DMKCT analysis with students' WoT through Didactical Design Research (DDR). Unlike previous studies that separately investigated teacher competence and students' cognition, this study examines how the integration of teachers' epistemic and interactional competencies affects students' mathematical thinking in learning fractions. The novelty of this study lies in the integration of DMKCT, WoT, and DDR frameworks to analyze the transformation of mathematics instruction at the elementary school level. Therefore, the research questions can be formulated as follows:

1. How are teachers' Didactic-Mathematical Knowledge and Competence (DMKCT) demonstrated during mathematics instruction?
2. How do students demonstrate their Ways of Thinking (WoT) during mathematics learning and problem solving?
3. How does the relationship between the teacher's DMKCT and students' WoT, revealed through the three stages of DDR, inform a revised didactical design for teaching fractions?

THEORETICAL FRAMEWORK

DMKCT views mathematics teacher competence not only as mastery of mathematical concepts but also as the ability to analyze students' thinking processes, select appropriate mathematical representations, construct meaningful learning interactions, and reflect on teaching practices (Anwar et al, 2025). DMKCT comprises three main dimensions: the mathematical dimension, the didactic dimension, and the meta-didactic-mathematical dimension. The mathematical dimension relates to mastery of mathematical concepts and procedures; the didactic dimension

concerns teaching strategies, student characteristics, learning interactions, and the use of learning media; and the meta-didactic dimension involves critical reflection and evaluation of teaching practice (Pereira & Kaiber, 2022). This study focuses on the didactic and mathematical dimensions.

Scheiner et al, (2023) explain that the transformation of teachers' mathematical knowledge is not only practical but also theoretical. They distinguish between mathematical knowledge for teaching and mathematics didactic knowledge. Mathematical knowledge for teaching emphasizes the use of mathematical knowledge in instructional practices, whereas mathematics didactic knowledge emphasizes the process of elementaryizing mathematical concepts to make them more accessible to students. The transformation of mathematical knowledge for teaching is also related to teachers' ability to connect formal mathematical concepts with the real-life learning contexts of elementary school students (Saadati et al., 2025). Therefore, teachers need to transform abstract concepts into concrete learning experiences through representations, manipulatives, visual illustrations, and exploratory activities (Newton et al., 2022). In this case, the transformation process does not merely simplify the material but also preserves its essential mathematical meaning.

Mathematical Way of Thinking (WoT) is a thinking ability that reflects how individuals understand, analyze, and solve mathematical problems logically, systematically, reflectively, and abstractly (Tallman, 2023). WoT is not only related to calculation skills but also involves reasoning, generalization, argumentation, representation, and mathematical modeling (Faizah et al., 2022). Students with good WoT are able to connect mathematical concepts to real-life situations, develop problem-solving strategies, and explain mathematical reasoning logically (Burgos & Godino, 2022). The DMKCT approach is closely related to WoT because one of the core competencies of mathematics teachers is understanding students' thought processes when solving mathematical problems. Teachers need to be able to analyze student strategies, identify conceptual errors, and build a learning environment that supports mathematical exploration and reflection

The quality of teachers' didactic-mathematical knowledge directly influences the development of students' mathematical thinking. Teachers with strong didactic-mathematical competence are able to design learning activities based on exploration, mathematical discussion, and problem solving, which encourage students to think critically and reflectively (Pereira & Kaiber, 2022; Yumiati et al., 2026). In the elementary school context, the relationship between the transformation of DMKCT and students' WoT becomes even more important because students are still developing the conceptual foundations of mathematics. When teachers are able to transform abstract concepts into concrete and meaningful activities, students can more easily develop conceptual understanding and deeper mathematical thinking patterns (Faizah et al., 2026; Sudirman et al., 2025). Conversely, overly procedural learning may limit the development of students' WoT. It has also been argued that a superficial understanding

of comparison among prospective teachers may be passed on to the students they teach (Irfan et al., 2018).

Recent research also shows that teachers who understand students' cognitive and affective dimensions are more effective in creating a mathematics learning environment that supports students' self-confidence, mathematical argumentation, and reflection. Thus, the transformation of DMKCT not only affects the quality of teaching but also contributes to the development of elementary school students' mathematical identity and culture of thinking.

METHOD

Research Design

This study employed Didactical Design Research (DDR) developed by Suryadi (2013, 2019a), a qualitative research design that is rooted in Brousseau's (1997) theory of didactical situations and has recently been applied in elementary school, junior high school, and teacher education contexts (Fauzi et al., 2023; Lumbantoruan, 2026; Yazidah & Sulistyorini, 2026). In this study, DDR does not refer to Design and Development Research (Richey & Klein, 2007), which is organized around needs analysis, the development of a product or model, and its evaluation. DDR instead centers on analyzing the didactical situation before, during, and after instruction, on anticipating and examining students' learning obstacles, and on refining a didactical design through repeated cycles. Following Suryadi (2019a), the study is grounded in two paradigms. The interpretive paradigm was used to understand how the teacher's didactical actions (DMKCT) and students' thinking (WoT) unfolded in the classroom, including the learning obstacles that students experienced. The critical paradigm was used to move from understanding to improvement, on the premise that no didactical design is ever complete and that every design must be revised on the basis of empirical evidence.

DDR consists of three interrelated stages (Suryadi, 2013, 2019b). The first stage is prospective analysis, which analyzes the didactical situation before instruction and results in a hypothetical didactical design, including the didactical and pedagogical anticipation (*antisipasi didaktis dan pedagogis*, ADP) of students' possible responses. In this study, the prospective analysis examined the mathematical content (equivalent fractions and the addition of fractions with different denominators), the teacher's planned didactical situation, and the potential learning obstacles of fifth-grade students, which were classified as ontogenic, epistemological, and didactical obstacles (Brousseau, 1997). The second stage is metapedadidactic analysis, which analyzes the didactical situation as it unfolds during instruction. It examines the teacher's ability to view the didactical relation between students and the material (HD), the pedagogical relation between the teacher and students (HP), and the ADP between the teacher and the material as a unified whole; to create didactical and pedagogical situations that meet students' needs; and to analyze students' responses and take follow-up actions accordingly. This ability is examined through three components, namely unity, flexibility, and coherence (Suryadi, 2013).

In this study, the epistemic and interactional aspects of the teacher's DMKCT and the students' WoT were analyzed within this stage.

The third stage is retrospective analysis, which relates the hypothetical didactical design from the prospective analysis to the results of the metapedadidactic analysis. In this study, it aimed to examine the correspondence between the planned and the enacted didactical situations and to formulate a revised (empirical) didactical design that can be refined further in subsequent DDR cycles. This article reports one DDR cycle, which ends with the proposal of a revised didactical design.

Data Collection

The participants in this study were one fifth-grade teacher and the students of the teacher's class from a public elementary school in Malang, East Java, Indonesia. The teacher selected as the research subject had more than five years of teaching experience. The selection of the teacher was based on the teacher's didactical and pedagogical competence in teaching mathematics. Meanwhile, the student participants were selected based on their ability to verbally explain their thinking processes when receiving information from the teacher during instruction and when solving mathematical problems.

Data were collected through observations, written tests, and interviews. Observations were conducted to examine the teacher's didactical and pedagogical competence in teaching mathematical concepts. The observations also served as a basis for identifying the relationship between didactical obstacles during instruction and students' WoT when solving mathematical problems. Written tests were used to identify students' ability to solve mathematical problems. Through these tests, the researcher identified students who experienced misconceptions and those who did not. Meanwhile, interviews were conducted to explore students' WoT in greater depth. Interviews were also used to identify students' obstacles or misconceptions during mathematics learning. In addition to interviewing students, the researcher also interviewed the teacher to examine the consistency between data obtained from the teacher and data obtained from the students. Following the DDR stages, the prospective analysis drew on the literature on fraction learning obstacles and on the teacher's planned instruction; the metapedadidactic analysis drew on the video-recorded observation, field notes, students' written tests, and interviews with the teacher and students; and the retrospective analysis drew on the comparison between the hypothetical didactical design and the results of the metapedadidactic analysis.

Data Analysis

The data analysis followed the three stages of DDR. In the prospective analysis, the didactical situation was analyzed to anticipate students' learning obstacles and to formulate a hypothetical didactical design with its ADP. In the metapedadidactic analysis, the observation data were analyzed in terms of the unity, flexibility, and coherence of the teacher's didactical and pedagogical actions, while the interview and written test data were analyzed to identify students' WoT and learning obstacles (ontogenic, epistemological, and didactical; Brousseau,

1997). In the retrospective analysis, the hypothetical didactical design was compared with the results of the metapedadidactic analysis to identify correspondences and discrepancies between the anticipated and the actual teacher actions and student responses, and these were used to formulate a revised didactical design. The field notes, transcripts, video recordings, and students' written work were coded and categorized according to the epistemic and interactional aspects of DMKCT and students' WoT. Observational data were analyzed by examining the teacher's learning process. During the observations, the researcher took field notes based on the epistemic and interactional aspects of the DMKCT. The learning process was also recorded to support a more detailed analysis of the teacher's DMKCT.

Classroom observation results were analyzed to identify the teacher's didactic practices, student responses, and possible didactic obstacles during the learning process. Students' written test results were analyzed to identify their Ways of Thinking (WoT), particularly in solving problems related to fractions. Interview data was used as a form of source triangulation to examine consistency between the teacher's teaching practices, student test results, and students' explanations of their thinking processes. Interpretation of the research findings was based on the relationship between the teacher's DMKCT and students' WoT during mathematics learning and problem-solving.

Data validity was ensured through triangulation, comparing data obtained from the teacher and students. The triangulation process involved not only interview data but also observational data from the learning process and students' written test results. Data inconsistent with the research objectives were further examined and considered potential findings.

RESULTS AND DISCUSSION

Results

The results are presented following the three stages of DDR. The prospective analysis presents the didactical situation, the anticipated learning obstacles, and the hypothetical didactical design that were established before the lesson. The metapedadidactic analysis presents the enacted lesson, namely the epistemic and interactional aspects of the teacher's DMKCT and the students' WoT that emerged during instruction. The retrospective analysis relates the two and formulates a revised didactical design.

Prospective Analysis: Didactical Situation, Anticipated Learning Obstacles, and Hypothetical Didactical Design

The prospective analysis was carried out before the lesson on fractions in the fifth-grade class. The mathematical content to be taught was equivalent fractions and the addition of fractions with different denominators, including in the form of story problems. Drawing on the literature on fraction learning (e.g., Debrenti, 2015; Sahidin et al., 2019; Tossavainen & Helenius, 2024) and on the teacher's planned instruction, three groups of potential learning obstacles were anticipated. Ontogenic obstacles were anticipated in students' readiness, for instance the

tendency to treat a fraction as two separate whole numbers and to associate the size of a representation with the size of the numerator or denominator. Epistemological obstacles were anticipated in the meaning of equivalence, which students may reduce to the rule of multiplying the numerator and denominator by the same number without understanding that the same whole is partitioned into different numbers of equal parts, and in the meaning of operations in story problems, which students may reduce to keywords. Didactical obstacles were anticipated to arise if instruction moved directly to procedures and keywords without visual representations and without discussion of students' ideas.

On this basis, a hypothetical didactical design was formulated. For equivalent fractions, the design anticipated a Concrete–Representational–Abstract (CRA) sequence: students would first compare fractions using concrete or pictorial models of the same whole partitioned into different numbers of equal parts, then represent the equivalence with diagrams or number lines, and only then formalize the rule of multiplying or dividing the numerator and denominator by the same number. The ADP for this part included questions that ask students to explain why the shaded area remains the same although the denominator changes, thereby addressing the anticipated misconception that a larger denominator implies a larger picture. For the addition of fractions with different denominators in story problems, the design anticipated that students would first retell the situation in their own words and represent it before choosing an operation, and that the teacher would follow up on students' interpretations rather than rely on keywords. Together, these anticipations formed the hypothetical didactical design against which the enacted lesson was analyzed.

Metapedadidactic Analysis: Classroom Interaction and Students' WoT

The metapedadidactic analysis was conducted by observing and exploring the enacted mathematics lesson in a fifth-grade elementary school classroom. The topics taught by the teacher were equivalent fractions and the addition of fractions with different denominators. The exploration was conducted not only with the teacher but also with the students, particularly when they completed the written test. The researcher observed the teacher's activities during instruction to identify the teacher's Didactic-Mathematical Knowledge and Competence for Teachers (DMKCT). After the learning process, the researcher conducted an interview with the teacher.

The observation of the enacted lesson showed that students experienced difficulties in understanding equivalent fractions conceptually. Most students were able to perform procedural calculations correctly; however, they struggled to represent fractions visually and explain the relationship between numerators and denominators. The teacher explained equivalent fractions procedurally by instructing students to multiply the numerator and denominator by the same number. During the interview, the teacher stated: "I usually teach equivalent fractions by showing students how to multiply both numbers using the same value because students understand faster that way*".* This finding demonstrates that instruction emphasized procedural fluency rather than conceptual understanding.

During classroom instruction, the teacher mainly used expository teaching methods and YouTube videos to explain fraction operations. Students had limited opportunities to discuss their reasoning or explore visual representations independently. The analysis of students' written work revealed misconceptions regarding the visual representation of equivalent fractions. One student represented fractions using unequal partition sizes, even though the fractions were mathematically equivalent.

During the interview, the student explained: *"*I thought that if the denominator becomes larger, then the picture should also become larger*".* This response indicates that the student associated denominator values with physical size rather than proportional relationships.

Viewed through the three components of metapedadidactic analysis, the enacted lesson showed the following pattern. In terms of unity, the teacher's actions concentrated on the relation between the teacher and the material, that is, on explaining rules and procedures, whereas the didactical relation between students and the material (opportunities to explore representations) and the pedagogical relation between the teacher and students (dialogue about students' ideas) were weakly developed; the three sides of the didactical triangle were therefore not treated as a whole. In terms of flexibility, when students showed confusion about visual representations and about choosing an operation in story problems, the teacher continued with direct explanation of the procedure rather than adjusting the didactical situation to students' responses. In terms of coherence, the didactical situation created at the beginning of the lesson, namely video-based examples followed by independent practice, was maintained throughout, and the changes in students' responses were not followed by didactical and pedagogical interventions that connected the context, the representation, and the operation. The epistemic and interactional aspects of the teacher's DMKCT, together with the students' WoT that emerged within this process, are elaborated in the following subsections.

DMKCT of Epistemic Aspect

In the epistemic aspect, the learning received by students tended to be unidirectional and mechanical. Students memorized solution steps rather than developing a deep understanding of mathematical concepts. The approach used by the teacher had not fully followed effective stages of fraction learning, such as the Concrete–Representational–Abstract (CRA) approach, which is particularly important in introducing fraction concepts. Ideally, the learning process should not start with formal rules such as multiplying or dividing the numerator and denominator, but with concrete and visual experiences that give these rules meaning (Tossavainen & Helenius, 2024). Teachers need to facilitate students with visual representations, such as pictures, diagrams, or number lines, so that students can build knowledge about the concept of fractions worth and the relationship between fractions. In this regard, the researcher (R) conducted an in-depth interview with the teacher (T) as in the following excerpt:

R: *What do you think about the concept of equivalent fractions and how to teach it to students?*

T: I think equivalent fractions are fractions that have the same value even though they are written in different fraction numbers. To explain to the students, I explained to students that a fraction worth is when the denominator and numerator are multiplied by the same number.
R: What obstacles do you think students experience when learning the concept of equivalent fractions?
T: In my opinion, students are still not able to describe equivalent fractions. If in a quick way, students understand more easily but if it is described, students are confused

The interview results show that the teacher's strategy in teaching equivalent fractions still focused on mechanical procedures and did not sufficiently strengthen students 'conceptual understanding. The teacher explained that equivalent fractions are obtained by multiplying the numerator and denominator by the same number. However, students were still confused when they were asked to represent equivalent fractions visually.

The teacher also realized that students tended to follow instructions without fully understanding the underlying concept. As a result, students experienced difficulty when solving problems independently. In addition, the teacher revealed that students understood equivalent fractions more quickly through direct calculation than through visual representation. This finding indicates that the teacher's approach emphasized procedural rules more than a deep understanding of equivalent fractions.

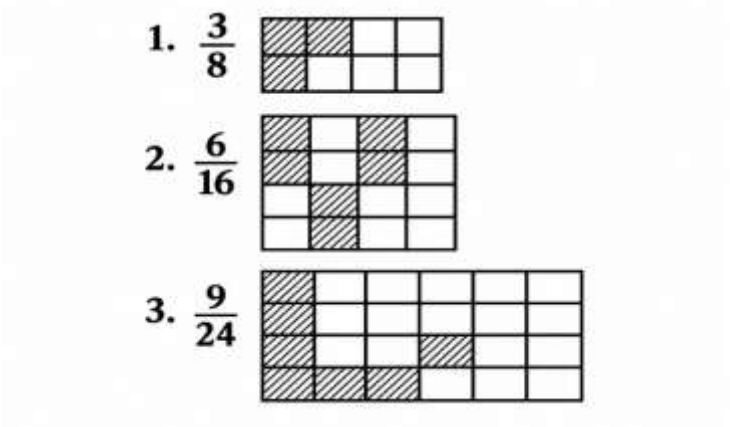
Students' WoT

Based on the teaching process carried out by the teacher, it can be seen that teaching mathematics on the concept of fractions does not have a significant impact on students' understanding. Figure 1 presents the results of students' work.

Question: Find two fractions that are equal to $\frac{3}{8}$ then draw all the fractions that you think are equal!

Figure 1.

Students' answers in visualizing the concept of equivalent fractions



The question asked students to find two fractions that are equal to $\frac{3}{8}$ and draw all the fractions that they considered equivalent. The results show that students experienced difficulty in representing equivalent fractions visually. Although students numerically understood that

equivalent fractions could be obtained by multiplying the numerator and denominator by the same number, errors occurred when they were asked to depict the fractions visually. The boxes used as representations were not divided proportionally; therefore, the shaded parts did not always represent equivalent fractions.

In addition, the division of the denominator in the fractions $\frac{6}{16}$ and $\frac{9}{24}$ did not seem to be in accordance with the principle that each part must still maintain equality with the initial fraction $\frac{3}{8}$. This shows that students focused more on calculation procedures than on understanding the concepts of area and proportion in equivalent fractions. Therefore, a more systematic visual approach is needed in fraction learning. To explore the problem, the researcher (R) conducted an in-depth interview with the student (S) as shown in the following transcript:

R: Why did you perform the operation 8×3 *to get a denominator of 24?*
S: Because I also multiplied the numerator by 3.
R: Do you think if the fractions are equal then the size of the pictures should be the same?
S: No, because the larger the denominator, the larger the image

The student's answer shows a misconception in understanding the concept of equivalent fractions, especially in relation to the size of the visual representation. The student assumed that the larger the denominator of a fraction, the larger the size of the picture should be. In fact, the denominator indicates the number of equal parts into which a whole is divided. Therefore, the larger the denominator, the smaller each part becomes, provided that the whole remains the same.

Table 1.
Epistemic Aspect of DMKCT and Students' WoT

Epistemic Aspect of DMKCT	Students' WoT
The teacher demonstrated knowledge of mathematical ideas and concepts.	Students experienced misconceptions due to errors in constructing visual representations.
The teaching process focused on procedural steps without sufficiently strengthening students' conceptual understanding.	Students were not fully able to think formally to understand the concept of fractions worth Concrete Representation Abstract (CRA).
This aspect is closely related to students' cognition in understanding mathematical concepts.	

This response shows that the student had not fully understood the concept of equivalent fractions. The student recognized that the numerator and denominator could be multiplied by the same number but did not understand that equivalent fractions represent the same value even though they are written with different denominators. Visually, equivalent fractions should represent the same overall quantity, although the whole may be divided into smaller equal parts

according to the denominator. In general, the teacher's didactical obstacles in the epistemic aspect and students' Ways of Thinking (WoT) are presented in Table 1.

DMKCT of Interactional Aspect

The interactional aspect refers to teachers' knowledge of the interactions that occur in the classroom. It includes the knowledge needed to predict, implement, and evaluate interactions among participants in the teaching and learning process. These interactions are oriented toward construction, negotiation, and clarification of learning meanings. These interactions do not only occur between teachers and students (teacher-student), but can also occur between students (student-student).

During the teaching process, the teacher did not provide sufficient space for students to explore or discuss their initial understanding. Instead, the teacher presented the material directly through an expository approach. At the beginning of the lesson, the teacher showed an example of a story problem from YouTube related to the addition of fractions with different denominators. The problem was as follows: "At the market, Ani bought flour and sugar to make a cake. She bought $\frac{2}{3}$ kg of flour and $\frac{1}{4}$ kg of sugar. What is the total weight of the flour and sugar that Ani bought?"

After showing the problem, the teacher immediately asked students about the calculation operation needed to solve it. However, several students seemed confused when determining the type of operation that should be applied. They had difficulty transforming the verbal information in the story problem into mathematical sentences.

The teacher did not explore students' understanding in depth or provide gradual guidance. Instead, the teacher directly explained that the two fractions in the problem had to be added to obtain total weight. This approach shows the teacher's tendency to prescribe problem-solving procedures without first building students' conceptual understanding of the relationship between the problem context and the corresponding mathematical operation. After discussing one example problem, the teacher continued with five similar practice problems, all taken from YouTube learning videos. These problems focused on the addition of fractions with different denominators. Students were given 15 minutes to solve the problems independently.

During the problem-solving process, many students appeared confused because they were unable to understand the information presented in the YouTube video and had difficulty converting the story problem into mathematical sentences. This finding indicates that the teacher-centered learning approach and the limited interaction between the teacher and students restricted students' opportunities to construct meaning between the problem context and the required mathematical representation.

To explore the teacher-centered learning strategy further, the researcher (R) conducted an interview with the teacher (T), as shown in the following excerpt:

R: *How do you teach the concept of addition of fractions to students if it is a story problem?*

T: I started by giving practice problems sourced from a YouTube video. The video contains 5 problems but I made problem number 1 as an example to be discussed together. The problem contains the addition of fractions with different denominators. I read the content of the question and explained that the calculation operation used was addition because there was the word "total" in the question sentence. I instructed the students to equalize the denominators by determining the LMM first and then adding them up to find the result.
R: Do you feel that students really understand the concept of fraction addition?
T: Yes. After I explained the calculation operation in the form of a story problem, I gave students additional practice problems to check their understanding. However, many students still had difficulty applying the concept of fraction addition to story problems. Although they could calculate fractions in numerical form correctly, they became confused when faced with story problems. This is because they had to understand the information presented and translate it into mathematical sentences. Their main difficulty was determining the appropriate operation.

Based on the interview results, it can be interpreted that the learning strategy used to teach fraction addition in story problems was still teacher-centered and procedural. The teacher began the lesson by showing a YouTube video containing practice problems and then directly explained the steps for solving them without giving students sufficient space to explore their own thinking. This approach shows that the learning process is focused on providing algorithms or quick ways to solve problems, such as equalizing the denominator through the Least Multiple of Multiplication (LMM) then adding the numerators.

The teacher's statement also shows that students' conceptual understanding had not been developed optimally, especially in translating verbal contexts in story problems into mathematical forms. The teacher acknowledged that although students could perform fraction calculations, they still had difficulty identifying important information in story problems and determining the appropriate mathematical operation. This indicates that the learning process was procedural because it prioritized final answers rather than students' thinking processes. As a result, students' mathematical understanding was not constructed meaningfully.

Furthermore, the teacher stated that many students still depended on teacher assistance, such as asking again about the operation to be used or the next step in the solution. This shows that students' learning independence had not yet developed. The lack of dialogic learning strategies, student reasoning, and opportunities for students to construct their own understanding may have contributed to students' misconceptions and their limited ability to solve contextual problems.

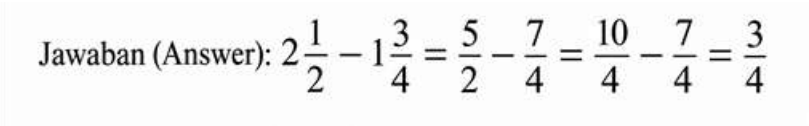
Students' WoT

Inappropriate teaching strategies may affect students' Ways of Thinking (WoT) when solving fraction addition problems with different denominators, especially when the problems are presented in the form of story problems. For example, students had difficulty determining the appropriate mathematical operation for adding fractions with different denominators.

Question: Someone used $2\frac{1}{2}$ cups of flour for a loaf of bread and $1\frac{3}{4}$ cups of flour to make cookies. How many cups of flour did they use in total for both foods?

Figure 2.

Students' Answer



Based on the analysis of the student's answer, a misconception was identified that reflects weaknesses in understanding fraction operations in the context of story problems. The student was asked to calculate the total amount of flour used to make two types of food, namely bread and cookies, which required $2\frac{1}{2}$ cups and $1\frac{3}{4}$ cups of flour respectively. The keyword "total" should be understood as a clue to perform the addition operation. However, students instead used the subtraction operation written as "$2\frac{1}{2} - 1\frac{3}{4}$ " which. This indicates that the student was unable to connect the contextual information in the problem with the appropriate mathematical operation.

This finding shows that the student did not yet have an adequate conceptual understanding of the meaning of addition in story problems. In addition, the error suggests that the student tended to focus on the numbers presented in the problem without fully understanding the meaning or context of the situation. To further explore the students' WoT in determining the mathematical operation used, the researcher (R) conducted an interview with the student (S), as shown in the following transcript:

R: *How did you understand this question when you first read it?*

S: When I read it, I saw that there were two numbers $2\frac{1}{2}$ *and* $1\frac{3}{4}$ *then I thought what was asked was the difference, so I used subtraction.*

R: What made you choose to subtract the two numbers?

S: Because I wanted to know how much flour was used for the bread. I didn't notice the word "total"

R: What do you think the word "total" in the question means?

S: I just realized that the word "total" means to add. So, it should add up all the flour

There were two conceptual and procedural errors that led to the selection of inappropriate mathematical operations. In the first problem, the keyword "total" indicated addition, but the student chose the subtraction operation between $2\frac{1}{2}$ glasses and $1\frac{3}{4}$. This error reflects a lack of conceptual understanding of the meaning of the word "total", which should indicate an accumulation or summation of values, not a subtraction operation. In the second l question about pizza, the phrase "$\frac{2}{3}$ part of $1\frac{1}{3}$ pizza" should be interpreted as a part of

the whole through a multiplication operation, which is then followed by a subtraction operation to determine the remainder. However, students instead added fractions that did not fit the context of the question. This shows that the student's semantic understanding of operational terms in the problem text had not yet developed optimally.

Mathematics learning theory emphasizes that conceptual understanding is the main foundation for solving story problems correctly. Sahidin et al. (2019) highlighted the importance of conceptual mastery so that students do not merely apply calculation procedures mechanically, but are also able to interpret the context of the problem and select the relevant mathematical operation. In general, the didactical obstacles in the interactional aspect and students' WoT are presented in Table 2.

Table 2.

Interactional Aspect and Students' WoT

DMKCT	Students' WoT
The learning process was teacher-centered, resulting in limited interaction between the teacher and students, as well as among students. The teacher did not provide sufficient space for students to explore their thinking because the learning process was teacher-centered and procedural. Teaching was conducted without sufficient conceptual exploration.	Students were unable to identify the appropriate mathematical operation presented in the story problem. This finding indicates that students did not have adequate conceptual understanding related to the meaning of addition operations in the context of fractions in story problems. The absence of a conceptual approach contributed to students' misconceptions when solving story problems.

Retrospective Analysis: Transformation of Instruction and Students' Thinking

The retrospective analysis compared the hypothetical didactical design with the enacted lesson and identified three discrepancies. First, the hypothetical design anticipated a Concrete–Representational–Abstract sequence for equivalent fractions, whereas the enacted lesson began with, and remained centered on, the rule of multiplying the numerator and denominator by the same number, so that visual representations were used only to a limited extent. Second, the anticipated student responses were largely confirmed: students associated the size of the picture with the size of the denominator, partitioned the whole unequally, and relied on keywords when choosing an operation. In addition, students' errors in story problems were more varied than anticipated, including choosing subtraction where addition was required and adding where multiplication followed by subtraction was required. Third, the anticipated pedagogical follow-up, in which the teacher builds on students' interpretations, was not enacted because the lesson remained expository and interaction was limited.

Based on Brousseau's (1997) classification, the association between the size of the picture and the size of the denominator, together with the understanding of equivalence as a rule only, can be regarded as epistemological obstacles, whereas the procedure-first and keyword-based instruction can be regarded as a didactical obstacle. These obstacles help explain why students who could perform the calculation correctly still failed to represent equivalent fractions visually and to select the appropriate operation in story problems.

The retrospective analysis showed that teacher-centered instruction influenced students' Ways of Thinking (WoT) by encouraging procedural reasoning rather than conceptual reasoning. Students relied heavily on keywords and memorized procedures when solving contextual fraction problems. The findings also demonstrated that visual scaffolding and conceptual representations were not sufficiently integrated into classroom instruction. Consequently, students experienced misconceptions in determining mathematical operations and interpreting contextual information.

Based on the results and discussion, the findings indicate that in the epistemic aspect, students tended to demonstrate more formal mathematical thinking because the teacher introduced the concept of equivalent fractions together with its symbolic rule, although visual representations were only limitedly integrated. In this aspect, students were able to use the rule conveyed by the teacher to solve numerical problems, but not to represent equivalent fractions visually.

Table 3.

Aspects of DMKC and Students' WoT

Aspect of DMKCT	Students' WoT	Description
Epistemic	Formal	The teacher introduced mathematical concepts and the multiplication rule, with limited visual representations. Students used the rule to solve numerical problems but had difficulty representing equivalent fractions visually.
Interactional	Non-Formal	The teacher experienced didactical obstacles in teaching mathematical concepts because the instruction relied heavily on videos and provided limited direct explanation and classroom interaction. Students made errors when solving mathematical problems due to misconceptions in understanding the problem context.

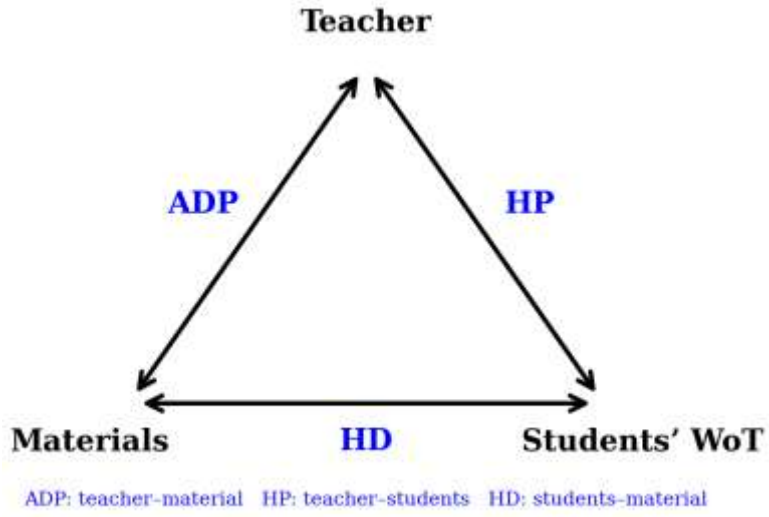
Meanwhile, in the interactional aspect, students tended to demonstrate non-formal mathematical thinking because the instruction was less conceptual and provided limited opportunities for interaction. In this case, the teacher mainly used learning videos from YouTube and did not sufficiently facilitate direct explanation, discussion, or exploration of students' reasoning. As a result, students experienced misconceptions when solving contextual

mathematical problems. Based on the research results, it was found that the learning design carried out by teachers consisted of aspects of DMKCT and students' way of thinking as presented in Table 3.

From Table 3, a prototype of the learning design is obtained which is interrelated between the teacher's didactics, students' WoT, and the fraction material being taught by the teacher in class as in Figure 3. The three sides of the triangle correspond to the modified didactical triangle in DDR, namely the didactical relation (HD) between students and the material, the pedagogical relation (HP) between the teacher and students, and the didactical and pedagogical anticipation (ADP) between the teacher and the material (Suryadi, 2013).

Figure 3.

Teachers' Didactic and Students' WoT within the Modified Didactical Triangle (HD, HP, and ADP)



On the basis of the retrospective analysis, the didactical design was revised into an empirical didactical design that is proposed for the next DDR cycle. For equivalent fractions, the revised design begins with a concrete or pictorial task in which students compare fraction strips or area models of the same whole, so that they can see that different fractions can occupy the same part of the whole. The teacher then asks students to explain why the shaded area does not change when the denominator changes, which directly confronts the idea that a larger denominator means a larger picture. Number lines and drawings serve as the representational step, and only after that are the rule of multiplying the numerator and denominator by the same number and its generalization introduced as the abstract step.

For the addition of fractions with different denominators, the revised design replaces the direct explanation of keywords with a sequence in which students first retell the story problem, draw or model the situation, and discuss which operation the situation requires, after which the teacher formalizes the procedure of finding a common denominator. The pedagogical anticipation for the teacher includes prompting questions (for example, what is being combined in this situation), inviting students to compare their solutions, and adapting the lesson when students' responses deviate from those anticipated, which strengthens the unity, flexibility, and

coherence of the didactical situation. Because this revised design has not yet been implemented, it should be treated as a hypothetical design for the second DDR cycle.

DISCUSSION

The results showed that the teaching strategy applied by the teacher was still oriented toward a procedural approach. The instruction relied mainly on direct explanations of multiplying or dividing the numerator and denominator by the same number. This approach provided limited opportunities for students to build conceptual understanding through concrete or visual representations. As a result, students' understanding of equivalent fractions tended to be shallow and focused mainly on technical procedures (Tossavainen & Helenius, 2024).

From a learning theory perspective, constructivism emphasizes that students should actively construct their own knowledge through interaction with the learning environment. Constructivist theory also suggests that concrete and visual representations are important elements that help students connect abstract concepts with real-life experience (Chai & Abdul Karim, 2023). Thus, an overemphasis on procedural methods in teaching equivalent fractions may ignore the role of interactive and contextualized learning experiences, resulting in limited conceptual understanding (Ruk & Zoest, 2023).

Furthermore, the procedural approach observed in this study may create a didactical obstacle, as students may find it difficult to reflect on or apply the concept of equivalent fractions in different situations because they have not understood the meaning behind the procedures taught (de Ron et al., 2022).In contrast, learning strategies that integrate innovative learning media, visual representations, and constructivist approaches can encourage students to actively explore fraction concepts in greater depth. This is in line with previous findings showing that the integration of visual and concrete approaches in mathematics learning can significantly improve students' conceptual understanding and application skills (Yuswan& Maat, 2021). Similarly, a quasi-experimental study in secondary mathematics found that differentiated instruction using tiered tasks, varied representations, and formative assessment produced significantly greater achievement gains than conventional lecture-based instruction (Abiodun et al., 2026). Technology-based media such as augmented reality have also been perceived by students and a teacher as helpful for visualizing geometry concepts, although constraints such as device requirements and internet quota need to be anticipated (Sudirman et al., 2021).

In epistemic aspect, teachers demonstrated knowledge of the mathematical ideas to be taught (Pino-Fan & Godino, 2011). In addition, teachers also need didactic-mathematical knowledge that includes various aspects or dimensions involved in the learning process, particularly the teacher's epistemic aspect and students' cognitive aspect, including knowledge of how students learn mathematics (Godino et al., 2017).

Students' limited understanding of the relationship between fraction values and area representations is consistent with previous studies on the development of fraction sense. The concept of fractions is formed through various subconstructs, including visual understanding.

Therefore, consistent use of visual representations is important in helping students understand the relationship between parts and wholes in fractions. The systematic use of visual models allows students to see that although the denominators are different, the total area representing the fraction remains the same (Tossavainen & Helenius, 2024).

Students' difficulty in connecting visual representations with numerical meaning often indicates a limited understanding of proportion in fractions (Debrenti, 2015). This can be seen in the student's statement that "the larger the denominator is, the larger the picture should be." This way of thinking indicates that the student associated the physical size of the image with the value of the fraction, whereas the concept of equivalent fractions should be understood through the ratio or proportional relationship between the parts and the whole.

This indicates that although students may be able to obtain the correct numerical result, they do not necessarily understand the reasoning behind the procedure used in equivalent fractions. Students' success in obtaining the correct answer does not always reflect deep conceptual understanding; instead, it may result from procedural learning and pattern memorization (Sahidin et al., 2019). Students' Ways of Thinking can be identified when they interpret the information in a problem, determine the relevant mathematical concept, and apply it to solve the problem (Faizah et al., 2022).

The results of the study are in line with the opinion of Inglada et al. (2025) which states that the epistemic facet refers to specific knowledge about the dimensions of mathematics. Timely feedback and a balance between challenge and guidance have likewise been identified by instructors and students as key supports for learning abstract mathematics (Haji et al., 2026). Teachers need to understand how mathematical objects can be represented, how tasks can be solved through different procedures, and how mathematical objects are connected to other concepts taught at previous or subsequent levels. Meanwhile, the interactional aspect refers to knowledge about classroom interactions. This includes the knowledge needed to predict, implement, and evaluate interactions among participants in the teaching and learning process. These interactions are oriented toward the construction, clarification, and negotiation of learning meanings. Such interactions may occur not only between teachers and students but also among students.

Therefore, to overcome students' shallow understanding of mathematical concepts caused by procedure-oriented instruction, a shift in the teaching paradigm is needed. Teachers need to design learning processes that stimulate students' Ways of Thinking through visual representations and conceptual scaffolding. These strategies can help students understand equivalent fractions more comprehensively. This shift is consistent with previous studies emphasizing the importance of constructivist approaches in facilitating conceptual understanding in mathematics. Through constructivist learning, students are not only guided to follow technical steps but are also encouraged to understand the concepts, meanings, and procedures involved in mathematical problem solving (Chai & Abdul Karim, 2023; Sercenia et al., 2023).

From the perspective of DDR, these findings show that the teacher's DMKCT cannot be separated from the didactical situation in which students' WoT emerge. The metapedadidactic analysis showed that the gap between the anticipated and the enacted didactical situation lay mainly in the pedagogical relation and in the teacher's flexibility in responding to students' ideas, and it is this gap that the retrospective analysis addressed through the revised design. In line with the critical paradigm of DDR (Suryadi, 2019a), the revised design is not a final product but a hypothesis to be tested and refined in subsequent DDR cycles.

CONCLUSION

This study examined the transformation of Didactic-Mathematical Knowledge and Competence for Teachers (DMKCT) and students' Ways of Thinking (WoT) in elementary mathematics learning on fractions using the Didactical Design Research (DDR) framework. The findings showed that although the teacher demonstrated adequate epistemic knowledge of fraction concepts, the instructional practice remained predominantly procedural and teacher centered. Through the three stages of DDR, the study also produced a revised didactical design for teaching equivalent fractions and fraction addition, which emphasizes a Concrete–Representational–Abstract sequence and dialogic interaction.

In the epistemic aspect, the teacher introduced mathematical concepts and visual representations, but students still experienced misconceptions in visualizing equivalent fractions. This indicates that visual representation was not sufficiently integrated to support students' movement through the Concrete–Representational–Abstract (CRA) process. In the interactional aspect, limited classroom interaction and insufficient conceptual exploration reduced students' opportunities to construct mathematical understanding. As a result, students relied on keywords and memorized procedures rather than conceptual reasoning when solving contextual fraction problems.

The novelty of this study lies in the integration of DMKCT analysis, students' WoT, and DDR to examine the transformation of mathematics instruction in elementary classrooms. The study contributes theoretically by showing how the epistemic and interactional dimensions of DMKCT influence students' mathematical reasoning, conceptual understanding, and misconceptions in learning fractions. Practically, the findings suggest that mathematics teachers need to reduce the dominance of procedural teaching and provide more opportunities for students to explore, discuss, visualize, and construct mathematical meanings collaboratively.

This study has several limitations. First, the study involved only one teacher and one classroom context, limiting the generalizability of the findings. Second, the study focused only on fraction concepts at the elementary school level, so the findings may not fully represent students' WoT in other mathematical topics or grade levels. Third, this study reports only one DDR cycle, and the revised didactical design has not yet been implemented and tested; future DDR cycles should implement the revised design and repeat the prospective, metapedadidactic, and retrospective analyses. Future studies are recommended to involve larger participant

groups, different mathematical topics, and longitudinal instructional interventions to examine the long-term development of students' mathematical thinking. Practically, the findings suggest that mathematics teachers should integrate conceptual representations, visual scaffolding, and interactive learning environments to strengthen students' mathematical understanding and Ways of Thinking. Teachers are also encouraged to reduce procedural teaching dominance and provide opportunities for students to construct mathematical meaning collaboratively.

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